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REDUCTION THEOREMS FOR NETWORKS WITH GENERAL SEQUENCING RELATIONS*

by

M. GLINZ AND R. MÖHRING

INTRODUCTION

Networks with general sequencing relations are generalizations of the well known CPM and MPM networks, achieved by the introduction of four different types of sequencing relations between the starting and finishing times of two different activites.

The main topics investigated in this paper are the existence and characterization of two-step procedures, (i.e. evaluating the subnetworks of a decomposition of a given network and of the condensed network assigned to the decomposition) for computing the shortest overall duration and the floats of a given network N independent of the durationsof N. It is shown that if sequences are allowed maximum durations, in this general approach only trivial solutions are obtained which are of no use for practical application. The restriction to minimum durations gives better results, but compared with CPM networks (cf [8]) they are still rather limited. It must therefore be said that, compared with CPM networks, generalization of the network structure must be paid for with a considerable limitation of the available reduction possibilities.

1. DESCRIPTION OF THE MODEL

A network for project scheduling is characterized by

a) a finite set A with elements $\alpha, \beta, \ldots,$ called *activities* of the project

b) *activity durations* $x(\alpha)$ for each $\alpha \in A$ or, in the stochastic case, non-negative real random variables X_α, their realizations being possible durations of α

c) a *project structure:* this may be deterministic (fixed sequencing relations between each pair of activities) or stochastic (e.g. if there are alternative procedures for accomplishing a given aim).

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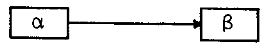
In the simplest case the durations $x(\alpha)$, $\alpha \in A$ are fixed real numbers and the project structure is given by a relation R in A defined by

$$(\alpha,\beta) \in R \quad :\Longleftrightarrow \quad \alpha \text{ must be finished before } \beta \text{ can be started.}$$

This network model forms the basis of the classical critical path method (CPM) and of the network theory developed in [2]. In the latter case it leads to very profound and far-reaching results, but it has, of course, the disadvantage of a rather inflexible project structure.

In the general case of a deterministic project structure all four possible sequencing relations between the starting and finishing times of two different activities are admitted ($ST(\alpha)$ representing the starting time and $FT(\alpha)$ the finishing time of α).

sequences (seq.)			
sort	term	symbol	diagram 1)
$ST(\alpha) \leq ST(\beta)$	begin seq. BS	$(\alpha,\beta) \in R_b$	
$FT(\alpha) \leq FT(\beta)$	end seq. ES	$(\alpha,\beta) \in R_e$	
$FT(\alpha) \leq ST(\beta)$	normal seq. NS	$(\alpha,\beta) \in R_n$	
$ST(\alpha) \leq FT(\beta)$	jump seq. JS	$(\alpha,\beta) \in R_j$	

Furthermore, a minimum and a maximum duration can be assigned to each sequence. This is accomplished by two functions

$$\begin{aligned} d_i &: R_i \to R^1_+ \quad \text{(minimum duration)} \\ \overline{d}_i &: R_i \to \overline{R}^1_+ \quad \text{(maximum duration)} \end{aligned} \qquad d_i \leq \overline{d}_i \ , \quad i = b,e,n,j$$

and the definition

1) Each activity α is represented by a rectangle [α], its left (right) side denoting the begin (end) of α. The sequences are represented by the arrows between the corresponding sides of the rectangle.

$$(1.2)\begin{cases}(\alpha,\beta)\in R_b & :\Longleftrightarrow\ ST(\alpha)+d_b(\alpha,\beta)\leq ST(\beta)\leq ST(\alpha)+\overline{d}_b(\alpha,\beta)\\(\alpha,\beta)\in R_e & :\Longleftrightarrow\ FT(\alpha)+d_e(\alpha,\beta)\leq FT(\beta)\leq FT(\alpha)+\overline{d}_e(\alpha,\beta)\\(\alpha,\beta)\in R_n & :\Longleftrightarrow\ FT(\alpha)+d_n(\alpha,\beta)\leq ST(\beta)\leq FT(\alpha)+\overline{d}_n(\alpha,\beta)\\(\alpha,\beta)\in R_j & :\Longleftrightarrow\ ST(\alpha)+d_j(\alpha,\beta)\leq FT(\beta)\leq ST(\alpha)+\overline{d}_j(\alpha,\beta)\end{cases}$$

As is well known, the maximum duration t of a sequence $(\alpha,\beta)\in R_i$ can be interpreted as the negative minimum duration -t of the sequence $(\beta,\alpha)\in R_k$ [2], where k is given by the table

i	b	e	n	j
k	b	e	j	n

This becomes clear (for i = b) because of

$$\overline{d}_b(\alpha,\beta)=t \overset{(1.2)}{\Longleftrightarrow} ST(\beta)\leq ST(\alpha)+t$$
$$\Longleftrightarrow ST(\beta)-t\leq ST(\alpha)\overset{(1.2)}{\Longleftrightarrow} d_b(\beta,\alpha)=-t\,.$$

Sequence durations can thus be described by a single function $d_i\colon R_i\to R^1$. This leads to the following definition.

DEFINITION 1: A *network with general sequences* (shortened in the following to simply *network*) is a tuple $N=(A,x,(R_i,d_i)_{i\in I})$, $I=\{b,e,n,j\}$, where

- A is a finite set of elements $\alpha,\beta,\ldots$, called *activities*
- $x\colon A\to R^1_+$ is the *activity duration function*; $x(\alpha)$ is called the *duration of* α
- $R_i\subset A\times A$ for each $i\in I$. $(\alpha,\beta)\in R_i$ is called *sequence of sort i*, i.e. *begin seq.* if i = b, *end seq.* if i = e, *normal seq.* if i = n, *jump seq.* if i = j
- $d_i\colon R_i\to R^1$ for each $i\in I$. $d_i(\alpha,\beta)$ is called the *duration of the sequence* $(\alpha,\beta)\in R_i$.

β is called a successor [predecessor] of α if $(\alpha,\beta)\in R_i$ [$(\beta,\alpha)\in R_i$] for some $i\in I$.

2) If $(\alpha,\beta)\in R_k$ is already given, then the minimum duration $d_k(\beta,\alpha)$ must be formally replaced by $\max\,(d_k(\beta,\alpha),-t)$. These cases make no sense, however, and do not occur in practice.

The relational system $\mathscr{R}:=(A,(R_i)_{i\in I})$ is called the *structure of N*. $(x,d):=(x,(d_i)_{i\in I})$ is called the *duration function of N*. N is also denoted by $N=(\mathscr{R},x,d)$. If $B\subset A$, we call $\mathscr{R}|B:=(B,(R_i|B)_{i\in I})$, $R_i|B:=R_i\cap B\times B$, the *sub-system of* $\mathscr{R}$ *generated by B*. $N|B:=(\mathscr{R}|B,x,d):=(B,x|B,(R_i|B,d_i|B)_{i\in I})$ is called the *sub-network of N generated by B*.

We shall apply this network model to projects, which fulfil the following procedural conditions.

(1.3)

- (i) All activities start at fixed times i.e. $ST(\alpha)\geq 0\quad\forall\,\alpha\in A$
- (ii) A started activity proceeds without interruption and is finished after $x(\alpha)$ time units i.e. $FT(\alpha)=ST(\alpha)+x(\alpha)$
- (iii) For each $(\alpha,\beta)\in\bigcup_{i\in I}R_i$ we have
  $ST(\alpha)+d_b(\alpha,\beta)\leq ST(\beta)$ if i = b
  $FT(\alpha)+d_e(\alpha,\beta)\leq FT(\beta)$ if i = e
  $FT(\alpha)+d_n(\alpha,\beta)\leq ST(\beta)$ if i = n
  $ST(\alpha)+d_j(\alpha,\beta)\leq FT(\beta)$ if i = j
- (iv) The project is finished when all activities are finished, i.e. at the time $\max_{\alpha\in A}FT(\alpha)$.

These conditions are combined in the definition of a schedule.

DEFINITION 2: Let $N=(\mathscr{R},x,d)$ be a network.

a) $T\colon A\to R^1_+$ is called a *schedule for N* [3] if for all $\beta\in A$

$$T(\beta)\geq\begin{cases}\max\limits_{(\alpha,\beta)\in R_b}{}^{4)}[T(\alpha)+d_b(\alpha,\beta)]\\\max\limits_{(\alpha,\beta)\in R_e}[T(\alpha)+x(\alpha)+d_e(\alpha,\beta)-x(\beta)]\\\max\limits_{(\alpha,\beta)\in R_n}[T(\alpha)+x(\alpha)+d_n(\alpha,\beta)]\\\max\limits_{(\alpha,\beta)\in R_j}[T(\alpha)+d_j(\alpha,\beta)-x(\beta)]\end{cases}$$

3) It should be noted that this notion of a schedule is only made possible by condition (1.3)(ii), which therefore cannot be replaced by the weaker condition $FT(\alpha)\geq ST(\alpha)+x(\alpha)$.

4) The maximum over the void set is defined as zero.

b) A duration function (x,d) on $\mathcal{R}$ is called *compatible* [5] (with respect to the given structure $\mathcal{R}$) if there exists a schedule for $(\mathcal{R},x,d)$. In this case, $(\mathcal{R},x,d)$ is called *practicable*.

c) Let T be a schedule for $N = (\mathcal{R},x,d)$. Then

$\lambda^T_{\mathcal{R}}(x,d) := \max_{\alpha\in A} [T(\alpha) + x(\alpha)]$ is called the *overall duration of N with regard to the schedule T*

$\lambda_{\mathcal{R}}(x,d) := \inf \{\lambda^T_{\mathcal{R}}(x,d) \mid T \text{ is a schedule for } N\}$ is called the *shortest overall duration* of N.

d) The shortest overall duration of a sub-network $N|B$ of N is denoted by $\lambda_{\mathcal{R}|B}(x,d)$

This concept of a network contains those of the classical CPM (only normal sequences of zero duration), the MPM (only begin sequences) and other related methods as special cases, cf. THUMB [13] for further information.

2. SHORTEST OVERALL DURATION, FLOATS

As there may not exist a schedule for a given network, the problem arises of finding criteria for the existence of a schedule, i.e. for the practicability of a network. A second task is the computation of the shortest overall duration.

These problems are solved by the same method as ROY [10] introduced for networks with begin sequences only (Metra-Potential Method).

To a given network $N = (\mathcal{R},x,d)$ we assign a digraph $G_N = (\mathcal{E},\mathcal{V})$ with edge values $y(e)$, $e \in \mathcal{E}$, as follows (cf. SCHWARZE, [11]).

5) It can be easily shown that incompatible duration functions can occur if the network structure contains a circuit.

(2.1)

The set $\mathcal{V}$ of the vertices of G_N is formed by doubling the activities $\alpha \in A$ into α^I (the initial vertex of α) and α^T (the terminal vertex of α).

The set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of the edges of G_N and their values are given by

(i) $\forall\, \alpha \in A$: $\quad (\alpha^I,\alpha^T) \in \mathcal{E} \wedge (\alpha^T,\alpha^I) \in \mathcal{E} \wedge y(\alpha^I,\alpha^T) = -y(\alpha^T,\alpha^I) = x(\alpha)$

(ii) $\forall\, (\alpha,\beta) \in R_b$: $\quad (\alpha^I,\beta^I) \in \mathcal{E} \wedge y(\alpha^I,\beta^I) = d_b(\alpha,\beta)$

$\forall\, (\alpha,\beta) \in R_e$: $\quad (\alpha^T,\beta^T) \in \mathcal{E} \wedge y(\alpha^T,\beta^T) = d_e(\alpha,\beta)$

$\forall\, (\alpha,\beta) \in R_n$: $\quad (\alpha^T,\beta^I) \in \mathcal{E} \wedge y(\alpha^T,\beta^I) = d_n(\alpha,\beta)$

$\forall\, (\alpha,\beta) \in R_j$: $\quad (\alpha^I,\beta^T) \in \mathcal{E} \wedge y(\alpha^I,\beta^T) = d_j(\alpha,\beta)$

G_N is called the *network graph of N.*

A *(directed) semi-path from $v_1 \in \mathcal{V}$ to $v'_n \in \mathcal{V}$* in G_N is a collection of edges $(v_1,v'_1),\dots,(v_n,v'_n)$ such that $v'_i = v_{i+1}$, $i = 1,\dots,n-1$. We denote it by the symbol $P(v_1,v'_n)$. A semi-path $P(v_1,v'_n)$ is called a *path from v_1 to v'_n* if all vertices v_i, v'_i of $P(v_1,v'_n)$ are distinct, and a *circuit* if $v'_n = v_1$.

$\sigma_{P(v_1,v'_n)}(y) := \sum_{j=1}^{n} y(v_1,v'_n)$ is called the *length* of $P(v_1,v'_n)$ (with respect to the edge values $y(v,v')$).

(2.2) EXAMPLE: Consider $N = (A,x,(R_i,d_i)_{i\in I})$ with

$\alpha \in A$	1	2	3	4	5
$x(\alpha)$	3	10	10	1	5

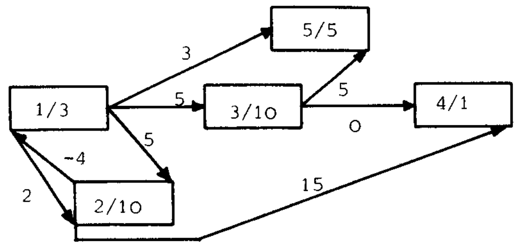
R_b	(1,2)	(2,1)
d_b	2	-4

R_e	(1,2)	(3,5)
d_e	5	5

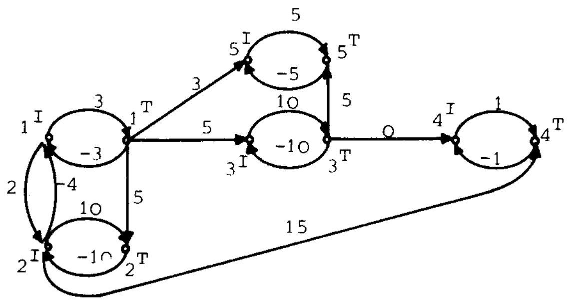
R_n	(1,3)	(1,5)	(3,4)
d_n	5	3	0

R_j	(2,4)
d_j	15

Diagram of N



The network graph G_N *of N*



Applying known results for potentials [6] on G_N we obtain the following theorem.

THEOREM 1:

a) *For a given network N there exists a schedule iff the associated network graph* G_N *contains no circuits of positive length.*

b) *If there exists a schedule for N, the shortest overall duration of N is equal to the length of the longest path of* G_N.

Proof: Let $N = (A,x,(R_i,d_i)_{i\in I})$ and $G_N = (\mathscr{V},\mathscr{E})$, with values of y as defined in (2.1).

6) See NEUMANN [6], p. 238 ff, or BERGE, [1] p. 89 ff for more detailed information.

(2.3) $\begin{cases}\text{Construct a digraph } G^\star = (\mathscr{V}^\star,\mathscr{E}^\star) \text{ from G by adding two} \\ \text{vertices a,b and all edges } (a,v),(v,b),\ v\in\mathscr{V}, \text{ with edge} \\ \text{values } y(a,v) = y(v,b) = 0.\end{cases}$

A real function τ defined on the vertices of $G^\star$ is called a *potential on* $G^\star$ if

$$\tau(v_2) \geq \tau(v_1) + y(v_1,v_2) \text{ for all } (v_1,v_2) \in \mathscr{E}^\star$$

We then obtain the following results

a) Let τ be a potential on $G^\star$, then $\tau(\alpha^T) = \tau(\alpha^I) + x(\alpha)$ for all $\alpha \in A$.

b) $\tau: \mathscr{V} \to R^1_+$ is a potential on $G^\star$ iff $T: A \to R^1_+$, defined by $T(\alpha) := \tau(\alpha^I)$, $\alpha \in A$, is a schedule for N.

For potentials, we have (cf. NEUMANN [6], p. 240 ff.) the following results.

(i) There exists a potential on $G^\star$ iff $G^\star$ contains no circuit of positive length.

(ii) If there exists a potential on $G^\star$ then $\tau_\star: \mathscr{V}^\star \to R^1$, defined by

$$\tau_\star(v) = \begin{cases} 0 & \text{if } v = a \\ \max\{\sigma_{P(a,v)}(y) \mid P(a,v) \text{ is a semi-path from a to v}\} & \text{if } v \neq a \end{cases}$$

is a non-negative potential on $G^\star$ and for any other potential τ on $G^\star$ with $\tau(a) = 0$ $\tau_\star \leq \tau$ holds. [7]

The theorem is proved by reformulating (i) and (ii) by using (a) and (b). In particular,

$$\begin{aligned}\tau_\star(b) &= \lambda_{\mathscr{R}}(x,d) \\ &= \max\{\sigma_{P(a,b)}(y) \mid P(a,b) \text{ is a semi-path from a to b}\} \\ &= \max\{\sigma_{P(a,b)}(y) \mid P(a,b) \text{ is a path from a to b}\},\end{aligned}$$

as $G^\star$ contains no circuit of positive length. ┘

7) This result assumes the existence of a singleton basis in the graph considered. In $G^\star$, {a} is that basis.

(2.4) REMARKS

a) The schedule assigned to $\tau_\star$ by (b) is denoted by ES. ES(α) is called the *earliest start of* α.

b) Using the same arguments as for (ii) it can be shown that

$$\tau^\star(v) := \begin{cases} \tau_\star(b) & \text{if } v = b \\ \tau_\star(b) - \max\{\sigma_{P(v,b)}(y) \mid P(v,b) \text{ is a path from } v \text{ to } b\} & \text{if } v \neq b \end{cases}$$

is a potential on $G^\star$ such that $\tau^\star \geq \tau$ for all potentials τ on $G^\star$ which fulfil $\tau(b) = \tau_\star(b)$.
$LF(\alpha) := \tau^\star(\alpha^T)$ is called the *latest finish of* α.

c) For later use, we define two related notions.
Let $G^\star - (v_1,v_2)$ denote the graph constructed from $G^\star$ by deleting the edge (v_1,v_2). Let $\tau_\star$ be the minimum potential (as defined in the proof of Theorem 1) on $G^\star - (a^T,a^I)$ and $\tau^\star$ the maximum potential (as defined in (2.4),b)) on $G^\star - (\alpha^I,\alpha^T)$. Then $ES^I(\alpha) := \tau_\star(\alpha^I)$ and $LF^T(\alpha) := \tau^\star(\alpha^T)$ are called the *earliest start of the initial point of* α and the *latest finish of the terminal point of* α respectively.

Next we shall state Theorem 1 without using the notion of a network graph.

DEFINITION 3: Let $N = (\mathcal{R},x,d)$ be a network and $R := \bigcup_{i\in I} R_i$.

a) $P(\alpha,\beta) := (\{\alpha_o = \alpha,\alpha_1,\dots,\alpha_n = \beta\},\mu)$ is called a *path in* $\mathcal{R}$ *from* α *to* β if $(\alpha_{j-1},\alpha_j) \in R$ for $j = 1,\dots,n$ and μ: $A\times A \to I$ is a mapping which assigns to each sequence (α_{j-1},α_j) its sort $\mu(\alpha_{j-1},\alpha_j)$.
A path $P(\alpha,\beta) = (\{\alpha_o,\dots,\alpha_n\},\mu)$ is called a *circuit in* $\mathcal{R}$ if $n \geq 1$ and $\alpha = \beta$.

b) Each path $P(\alpha,\beta) = (\{\alpha_o,\dots,\alpha_n\},\mu)$ corresponds to a semi-path $P(\alpha^I,\beta^T)$ from α^I to β^T in G_N that contains all edges of G_N assigned to the sequences $(\alpha_{j-1},\alpha_j) \in R_{\mu(\alpha_{j-1},\alpha_j)}$, $j = 1,\dots,n$.

$\sigma_{P(\alpha,\beta)}(x,d) := \sigma_{P(\alpha^I,\beta^T)}(y)$ is called the *length (duration) of* $P(\alpha,\beta)$ (with respect to the duration function (x,d) of N).

c) Similarly, each circuit $P(\alpha,\alpha)$ of $\mathcal{R}$ corresponds to a circuit P in G_N. $\sigma_{P(\alpha,\alpha)}(x,d) := \sigma_P(y)$ is called the *length (duration) of the circuit*.

Let $P(\alpha,\beta) = (\{\alpha_o,\dots,\alpha_n\},\mu)$ be a path. The corresponding semi-path P of G_N partitions the activities $\alpha_o,\dots,\alpha_n$ into the disjunct sets

$P^+ := \{\alpha_j \mid (\alpha^I,\alpha^T)$ is contained in P$\}$, the set of *positively traversed* activities,

$P^- := \{\alpha_j \mid (\alpha^T,\alpha^I)$ is contained in P$\}$, the set of *negatively traversed* activities and

$P^o := \{\alpha_o,\dots,\alpha_n\} \setminus (P^+ \cup P^-)$, the set of activities *touched by P*.

A sequence (α_o,β_o) is said to *lie on* $P(\alpha,\beta)$ or to *be traversed by* $P(\alpha,\beta)$ if the corresponding edge in G_N is contained in the semi-path of G_N corresponding to $P(\alpha,\beta)$. Then

$$\sigma_{P(\alpha,\beta)}(x,y) = \sum_{\alpha\in P^+} x(\alpha) - \sum_{\alpha\in P^-} x(\alpha) + \sum_{j=1}^{n} d_{\mu(\alpha_j,\alpha_{j+1})}(\alpha_j,\alpha_{j+1})$$

and we get:[8)]

THEOREM 1':

a) *For a given network N there exists a schedule iff N contains no circuits of positive length.*

b) *If there exists a schedule for N, then the shortest overall duration of N is equal to the length of a longest path in N.*

Each path $P(\alpha,\beta)$ of N, such that $\sigma_{P(\alpha,\beta)}(x,d) = \lambda_{\mathcal{R}}(x,d)$ is called *critical*. Its length is called the *critical length of N*.

(2.5) EXAMPLE (CONTINUATION OF (2.2))

The only circuit of $\mathcal{R}$ is $P(1,1) = (\{1,2,1\},\mu)$ with $\mu(1,2) = e$, $\mu(2,1) = b$. It corresponds to the circuit $(1^I,1^T),(1^T,2^T),(2^T,2^I),(2^I,1^I)$ of G_N. Its length is therefore -6 which means that there exists a schedule for N. T: $A \to R^1_+$, defined by

8) This follows immediately from the fact that to each circuit or longest path P of G_N there is a circuit or semi-path P' in N which corresponds to it in the sense of Definition 3.

α	1	2	3	4	5
T(α)	0	2	8	18	18

is a schedule and we have $T(\alpha) = ES(\alpha)$.
The critical path is $P(1,5) = (\{1,3,5\},\mu)$, $\mu(1,3) = n$, $\mu(3,5) = e$ with length 23. Thus $\lambda_{\mathscr{R}}(x,d) = 23$.

For the planning of a real project it is important to obtain (besides the shortest overall duration) some information on whether an activity can be extended or delayed as a whole without interfering (under certain conditions) with the shortest overall duration.

To this purpose, we introduce the three most frequently used float measures (without covering all possibilities).

DEFINITION 4: Let $N = (\mathscr{R},x,d)$ be a practicable network and $\alpha \in A$.

a) Let $EI_N^I(\alpha) := \min \begin{cases} \min\limits_{(\alpha,\beta)\in R_b}{}^{9)} [ES(\beta) - d_b(\alpha,\beta) + x(\alpha)] \\ \min\limits_{(\alpha,\beta)\in R_j} [ES(\beta) + x(\beta) - d_j(\alpha,\beta) + x(\alpha)] \end{cases}$

$EI_N^T(\alpha) := \min \begin{cases} \min\limits_{(\alpha,\beta)\in R_e} [ES(\beta) + x(\beta) - d_e(\alpha,\beta)] \\ \min\limits_{(\alpha,\beta)\in R_n} [ES(\beta) - d_n(\alpha,\beta)] \end{cases}$

$EI_N(\alpha) := \min\,[EI_N^I(\alpha), EI_N^T(\alpha)]$ is called the *earliest start of the (immediate) successors of* α.
It denotes the latest point of time at which α may be finished if all successors β of α may begin at their earliest start $ES(\beta)$ and $\lambda_{\mathscr{R}}(x,d)$ may not be enlarged.

b) Let $LI_N^I(\alpha) := \max \begin{cases} \max\limits_{(\beta,\alpha)\in R_b}{}^{10)} [LF(\beta) - x(\beta) + d_b(\beta,\alpha)] \\ \max\limits_{(\beta,\alpha)\in R_n} [LF(\beta) + d_n(\beta,\alpha)] \end{cases}$

$LI_N^T(\alpha) := \max \begin{cases} \max\limits_{(\beta,\alpha)\in R_e} [LF(\beta) + d_e(\beta,\alpha) - x(\alpha)] \\ \max\limits_{(\beta,\alpha)\in R_j} [LF(\beta) - x(\beta) + d_j(\beta,\alpha) - x(\alpha)] \end{cases}$

9) The minimum over the void set is defined to be $\lambda_{\mathscr{R}}(x,d)$.

10) The maximum over the void set is defined to be zero.

Then $LI_N(\alpha) := \max\,[LI_N^I(\alpha), LI_N^T(\alpha)]$ is called the *latest finish of the (immediate) predecessors of* α.
It denotes the earliest point of time at which α may start if all predecessors β may end at their latest finish $LF(\beta)$.
ES, LF, EI, LI are called *characteristic activity times.*
From them, the most important floats are derived.

c) $TF(\alpha) := LF(\alpha) - ES(\alpha) - x(\alpha)$ is called the *total float of* α
$FF(\alpha) := EI(\alpha) - ES(\alpha) - x(\alpha)$ is called the *free float of* α
$IF(\alpha) := EI(\alpha) - LI(\alpha) - x(\alpha)$ is called the *independent float of* α. 11)
Their interpretations are clear.

These floats describe the possibilities open for displacing an activity in time, i.e. with its duration unchanged. If, for example, the initial point of an activity α is touched by a critical path, α must start at ES(α) and any delay would affect the shortest overall duration. Thus no information is obtained if, with the starting time ST(α) fixed at ES(α), the finishing time FT(α) may be delayed (i.e. $FT(\alpha) > ES(\alpha) + x(\alpha)$, which means extending the duration of α).

This information gap is bridged by introducing the concept of extension floats 12), which are defined using ES^I, LF^T (cf. (2.4)) and EI^I, LI^T (cf. Definition 4).

DEFINITION: Let $N = (\mathscr{R},x,d)$ be a practicable network and $\alpha \in A$.
$TE_N^I(\alpha) := ES(\alpha) - ES^I(\alpha)$ is called the *total extension float at the initial point of* α.

$TE_N^T(\alpha) := LF^T(\alpha) - LF(\alpha)$ is called the *total extension float at the terminal point of* α.

$IE_N^I(\alpha) := LI(\alpha) - LI^I(\alpha)$ is called the *independent extension float at the initial point of* α.

$IE_N^T(\alpha) := EI^T(\alpha) - EI(\alpha)$ is called the *independent extension float at the terminal point of* α.

11) We have $TF(\alpha) > FF(\alpha) > \max(IF(\alpha),0)$. IF(α) can be negative and can then be interpreted as the time lacking to let all predecessors β of α end at LF(β) and all successors β' of α begin at ES(β') simultaneously.

12) cf SCHWARZE, [12]

It is now possible to specify the intervals of time from which ST(α) and FT(α) may be chosen, with the restriction, of course, that FT(α) ≥ ST(α) + x(α) and $\lambda_{\mathcal{R}}$(x,d) may not be lengthened. We obtain:

a) $ES(\alpha) - TE^{I}(\alpha) \leq ST(\alpha) \leq ES(\alpha) + TF(\alpha)$
$LF(\alpha) - TF(\alpha) \leq FT(\alpha) \leq LF(\alpha) + TF^{T}(\alpha)$

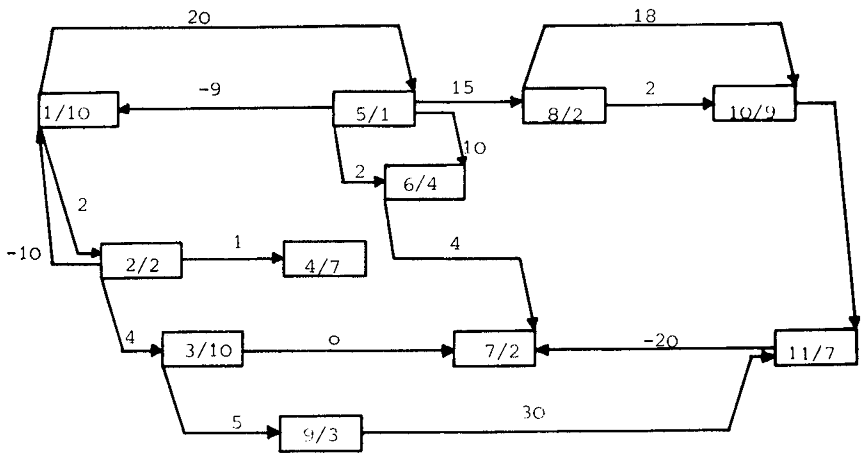
in the case that the choice of ST(α) and FT(α) may affect the starting and finishing times of other activities, and

b) $LI(\alpha) - IE^{I}(\alpha) \leq ST(\alpha) \leq EI^{I}(\alpha) - x(\alpha)$
$LI^{T}(\alpha) + x(\alpha) \leq FT(\alpha) \leq EI(\alpha) + IE^{T}(\alpha)$

if they may not.[13)]

(2.6) EXAMPLE:

Let N be given by the following diagram



13) If these inequalities are false, there is no such choice.

(Extension) floats and characteristic times of N are given by the following table:

α ∈ A	x	ES	LF	EI	LI	TF	FF	IF	TE^I	TE^T	IE^I	IE^T
1	10	0	10	10	0	0	0	0	0	47	0	47
2	2	2	10	4	2	6	0	0	0	39	0	0
3	10	6	22	16	12	6	0	-6	0	33	0	12
4	7	5	57	57	11	45	45	39	0	0	0	0
5	1	19	20	20	19	0	0	0	19	0	19	0
6	4	26	57	30	26	27	0	0	5	0	5	27
7	2	28	57	57	55	27	27	0	12	0	33	0
8	2	35	37	37	35	0	0	0	0	5	0	5
9	3	11	20	20	17	6	6	0	0	0	0	0
10	9	44	53	53	44	0	0	0	5	0	5	0
11	7	50	57	57	50	0	0	0	6	0	0	0

3. THE GENERAL REDUCTION PROBLEM

In the following paragraphs we shall deal with the problem of whether the shortest overall duration (and, if possible, floats and extensions) of a given network N can be computed by a two-step procedure in the following way.

N is partitioned into suitably defined sub-networks which correspond to exactly one activity in a smaller network, called the image network of N. Activity and sequence durations of the image network are derived from an evaluation of the corresponding sub--networks of N. Our question is, whether we can compute the desired quantities of N by evaluating the image network and the sub-networks only.

As a first approach to this problem we apply the results of a reduction theory for a general class of functions (network functions, see [2] and [4] [14]).

For a given network $N = (\mathscr{R},x,d)$, the paths from a to b in the graph $G^\star$ which is constructed from N according to (2.1) and (2.3) form a profile $P^\star$ on the edge set $\mathscr{E}^\star$ of $G^\star$. By Theorem 1, $\lambda_{\mathscr{R}}$ is a network function on $P^\star$, i.e.

$$\lambda_{\mathscr{R}}(x,d) = \Gamma[\tau,P^\star,\max,+](y)$$

where y is the edge value function of $G^\star$ derived from (x,d).

In general, however, there are only few reduction possibilities for Γ, as too many edges are added in constructing $G^\star$ from G. Confining ourselves to the system P of all paths of G_N we still have

$$\lambda_{\mathscr{R}}(x,d) = \Gamma[\tau,P,\max,+](y),$$

but P may not be a profile, since paths may be comparable under (set theoretical) inclusion.

Further restriction to $P^\circ := \{P \in P \mid \forall P' \in P: P \subset P' \Rightarrow P = P'\}$ yields no solution either, because $\subset$-maximal paths may not be the longest ones. Nevertheless, P° is of some importance. As P° is a profile, all its reduction possibilities can be characterized. (This is done in [3], where the profile P° plays a part in the theory of flows in networks). Because of $P^\circ \subset P$, all reduction possibilities of P° are contained in those of P.

The main disadvantage of this approach, however, is the fact that it covers only the shortest overall duration, and not floats and extensions. In order to include floats and extensions it is necessary to have a network structure (and not only a profile) on the image set. It should depend on the structure of N only, and not on the duration function (x,d) of N, for in this way reduction can be guaranteed in the stochastic case as well, i.e. for arbitrary durations. (This is the main difference to decomposition procedures described by PARIKH - JEWEL ([7]), LÜTGEN ([5]), REGITZ ([9]) e.a.)

In view of these considerations, the following approach is taken, which, no longer interpreting $\lambda_{\mathscr{R}}$ as a network function, resembles the poset-theory approach to the reduction problem for CPM networks (see [8]).

14) See also [2] for the definition of Γ, profile etc.

(3.1) CONSTRUCTION OF THE IMAGE NETWORK

Let $\mathscr{R} = (A,(R_i)_{i\in I})$ be a network structure. Given a partition $\pi = \{A_j \mid j \in J\}$ of A, let $h: A \to A' := \{\gamma_j \mid j \in J\}$ defined by $h(\alpha) := \gamma_j$ if $\alpha \in A_j$ denote the canonical mapping induced by π.
Let $\mathscr{R}' = (A',(R_i')_{i\in I})$ be a network structure on A'. To each image activity $\gamma_j \in A'$ we assign a real function $f_j \geq 0$ with variables $x(\alpha)$, $\alpha \in A$, and $d_i(\alpha,\beta)$, $(\alpha,\beta) \in R_i$, $i \in I$. Similarly, a real function g_{j,k,i_o} with the same variables as f_j is assigned to each image sequence (γ_j,γ_k) of sort i_o, $i_o \in I$.
f_j and g_{j,k,i_o} describe the durations of $\gamma_j \in A'$ and $(\gamma_j,\gamma_k) \in R'_{i_o}$ for given durations $x(\alpha)$, $\alpha \in A$, and $d_i(\alpha,\beta)$, $(\alpha,\beta) \in R_i$, $i \in I$, of N. They are called *transmitting rules*.

Our aim is to characterize all partitions π and their associated image structures $\mathscr{R}'$ and transmitting rules which satisfy

$\lambda_{\mathscr{R}}(x,d) = \lambda_{\mathscr{R}'}(x',d')$ for all compatible duration vectors (x,d) of N

under appropriate and comparatively weak conditions.
This leads to

(3.2) THE GENERAL REDUCTION PROBLEM

Let $\mathscr{R} = (A,(R_i)_{i\in I})$ be a network structure.
Characterize all partitions $\pi = \{A_j \mid j \in J\}$ of A, together with their assigned image structures $\mathscr{R}'$ and transmitting rules according to (3.1), which fulfil the following conditions:

R1 To each image sequence $(\gamma_j,\gamma_k) \in R_i'$, $i \in I$, there exist $\alpha \in A_j$, $\beta \in A_k$ such that $(\alpha,\beta) \in R_i$.
CONSERVATION OF STRUCTURE

R2 For each $j \in J$, f_j depends only on the variables $x(\alpha)$, $\alpha \in A_j$, and $d_i(\alpha,\beta)$, $(\alpha,\beta) \in R_i|A_j$, $i \in I$; i.e. $x'(\gamma_j)$ is a function of the durations of the activities and sequences in $\mathscr{R}|A_j$.
For each j,k,i_o, g_{j,k,i_o} depends only on the variables $d_{i_o}(\alpha,\beta)$, $(\alpha,\beta) \in R_{i_o} \cap A_j \times A_k$; i.e. $d'_{i_o}(\alpha,\beta)$ is a function of the durations of the sequences of sort i_o from A_j to A_k.
LOCAL TRANSMISSION [15]

15) This condition is essential for the implementation of a two-step computation on computers because of the saved storage.

R3 For each compatible duration function (x,d) on $\mathcal{R}$, (x',d') is a compatible duration function on $\mathcal{R}'$

COMPATIBLE TRANSMISSION

R4 For each compatible duration function (x,d) on $\mathcal{R}$ we have $\lambda_{\mathcal{R}}(x,d) = \lambda_{\mathcal{R}'}(x',d')$

CONSERVATION OF THE SHORTEST OVERALL DURATION

We shall show that R1 - R4 yield very restrictive, but necessary, conditions for partitions π (and the assigned image structure and transmitting rules), which limit the possibilities of reduction severely. The main reason for this limitation lies in the fact that sequences can have negative duration.

(3.3) $x \equiv 0,\ d_i \leq 0$ on $\mathcal{R}|A_j \Rightarrow x'(\gamma_j) = 0$
i.e. if all durations of the sub-network $\mathcal{R}|A_j$ are non-positive then the corresponding image activity γ_j is of duration zero.

Proof: Put $y(\alpha) := 0 \quad \forall\, \alpha \in A$, $l_i(\alpha,\beta) := \begin{cases} d_i(\alpha,\beta) & \alpha,\beta \in A \\ 0 & \text{otherwise} \end{cases}$

Obviously, (y,l) is compatible with respect to $\mathcal{R}$ and $\lambda_{\mathcal{R}'}(y',l') \overset{R4}{=} \lambda_{\mathcal{R}}(y,l) = 0$, as $(y,l) \leq 0$. Thus $y'(\gamma_j) = 0$. Because of the fact that $(y,l) = (x,d)$ on $\mathcal{R}|A_j$ and R2 we have $x'(\gamma_j) = y'(\gamma_j) = 0$. ┘

(3.4) $(\gamma_j,\gamma_k) \in R'_{i_o} \wedge d_i(\alpha,\beta) \leq 0 \quad \forall\, (\alpha,\beta) \in R_i \cap A_j \times A_k$
$\Rightarrow d'_{i_o}(\gamma_j,\gamma_k) \leq 0$
i.e. if the durations of all sequences on which an image sequence depends are non-positive, the image sequence has non-positive duration.

Proof: Let $(\gamma_j,\gamma_k) \in R_{i_o}$ and $d_i(\alpha,\beta) \leq 0$ for all $(\alpha,\beta) \in R_i \cap A_j \times A_k$.
Put $x \equiv 0$, $l_i(\alpha,\beta) := \begin{cases} d_i(\alpha,\beta) & \text{if } (\alpha,\beta) \in R_i \cap A_j \times A_k \text{ and } i = i_o \\ 0 & \text{otherwise} \end{cases}$
Then $(y,l) \leq 0$, is thus compatible, and we obtain
$\lambda_{\mathcal{R}'}(y',l') \overset{R4}{=} \lambda_{\mathcal{R}}(y,l) = 0$
From $\lambda_{\mathcal{R}'}(y',l') \geq l'_{i_o}(\gamma_j,\gamma_k)$ it follows that
$0 \geq l_{i_o}(\gamma_j,\gamma_k) \overset{R2}{=} d_{i_o}(\gamma_j,\gamma_k)$. ┘

(3.5) $x'(\gamma_j) = \lambda_{\mathcal{R}|A_j}(x,d)$
i.e. the duration of an image activity is equal to the shortest overall duration of the corresponding sub-network.

Proof: Given a compatible duration function (x,d), put
$y(\alpha) = \begin{cases} x(\alpha) & \alpha \in A_j \\ 0 & \text{otherwise} \end{cases}$, $d_i(\alpha,\beta) := \begin{cases} d_i(\alpha,\beta) & \text{if } \alpha,\beta \in A_j \\ -\lambda_{\mathcal{R}}(x,d) & \text{otherwise} \end{cases}$
As any circuit containing activities not in A_j must traverse a sequence of duration $-\lambda_{\mathcal{R}}(x,d)$, (y,l) is compatible, and is equal to (x,d) on A_j and ≤ 0 outside A_j. Therefore,
$\lambda_{\mathcal{R}|A_j}(x,d) = \lambda_{\mathcal{R}|A_j}(y,l) = \lambda_{\mathcal{R}}(y,l) \overset{R4}{=} \lambda_{\mathcal{R}'}(y',l')$
$\overset{(3.3),(3.4)}{=} y'(\gamma_j) \overset{R2}{=} x'(\gamma_j)$. ┘

(3.6) Let $(\gamma_j,\gamma_k) \in R'_{i_o}$. If there exists $(\alpha,\beta) \in R_{i_o} \cap A_j \times A_k$ such that $d_{i_o}(\alpha,\beta) > 0$, then $d'_{i_o}(\gamma_j,\gamma_k) = \max\{d_{i_o}(\alpha,\beta) \mid (\alpha,\beta) \in R_{i_o} \cap A_j \times A_k\}$

Proof: Given a compatible duration function (x,d), put $y \equiv 0$ and
$l_i(\alpha,\beta) := \begin{cases} d_i(\alpha,\beta) & \text{if } (\alpha,\beta) \in R_{i_o} \cap A_j \times A_k \\ -c & \text{otherwise} \end{cases}$
where $c := \max\{d_{i_o}(\alpha,\beta) \mid (\alpha,\beta) \in R_{i_o} \cap A_j \times A_k\} > 0$ according to the assumptions.
(y,l) is compatible, since each circuit traversing a sequence of positive duration ($\leq c$) must contain a sequence of duration $-c$.
Hence,

$$\begin{aligned} c &= \max\{d_{i_o}(\alpha,\beta) \mid (\alpha,\beta) \in R_{i_o} \cap A_j \times A_k\} \\ &= \max\{l_{i_o}(\alpha,\beta) \mid (\alpha,\beta) \in R_{i_o} \cap A_j \times A_k\} \\ &= \lambda_{\mathcal{R}}(y,l), \quad \text{as } c > 0 \text{ and any longest path consists of one sequence } (\alpha,\beta) \in R_{i_o} \cap A_j \times A_k \text{ only} \\ &\overset{R4}{=} \lambda_{\mathcal{R}'}(y',l') = l'_{i_o}(\gamma_j,\gamma_k) \quad \text{because of (3.3), (3.4) and } \lambda_{\mathcal{R}'}(y',l') > 0 \\ &= d'_{i_o}(\gamma_j,\gamma_k). \end{aligned}$$
┘

(3.7) $(\alpha,\beta) \in R_i \wedge h(\alpha) \neq h(\beta) \Rightarrow (h(\alpha),h(\beta)) \in R'_i$

Proof: Let $\alpha_o \in A_j$, $\beta_o \in A_k$, $j \neq k$, and $(\alpha_o,\beta_o) \in R_{i_o}$.
Assume $(\gamma_j,\gamma_k) \notin R'_{i_o}$. Let

$$x \equiv 0,\ d_i(\alpha,\beta) := \begin{cases} 1 & \text{if } (\alpha,\beta) = (\alpha_o,\beta_o) \wedge i = i_o \\ -2 & \text{otherwise} \end{cases}$$

(x,d) is compatible, as $(\mathcal{R},x,d)$ cannot contain a circuit of positive length. Because of $(\gamma_j,\gamma_k) \notin R'$ and (3.3),(3.4) we obtain $(x',d') \leq 0$, and so

$$0 = \lambda_{\mathcal{R}'}(x',d') \overset{R4}{=} \lambda_{\mathcal{R}}(x,d) \geq d_{i_o}(\alpha_o,\beta_o) = 1, \text{ a contradiction.}$$ ┘

Combining (3.3) - (3.7), we obtain the following conditions for the image structure $\mathcal{R}'$ and the transmitting rules.

THEOREM 2: *Let $\pi = \{A_j \mid j \in J\}$ be a solution of the general reduction problem (3.2). Then the assigned image structure and transmitting rules must fulfil the following conditions.*

a) *For all $\alpha,\beta \in A$ such that $h(\alpha) \neq h(\beta)$, and all $i \in I$, we have $(\alpha,\beta) \in R_i \longleftrightarrow (h(\alpha),h(\beta)) \in R'_i$*

b) *$x'(\gamma_j) = \lambda_{\mathcal{R}|A_j}(x,d)$ for all $\gamma_j \in A'$*

c) *$d'_i(\gamma_j,\gamma_k) = \max\{d_i(\alpha,\beta) \mid (\alpha,\beta) \in R_i \cap A_j \times A_k\}$ for all $(\gamma_j,\gamma_k) \in R'_i$, $i \in I$ for which the maximum on the right side is positive.*

These conditions can be considered natural, especially as they also result in a similar form of the reduction problem for CPM-networks (see [8]).

The main restriction of the reduction possibilities derives from the special form of the classes A_j of π.

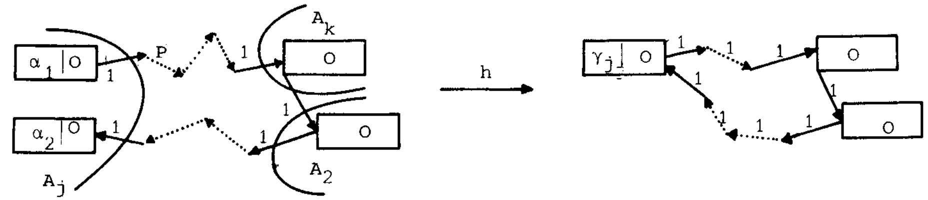
(3.8) Each path from $\alpha_1 \in A_j$ to $\alpha_2 \in A_j$, $\alpha_1 \neq \alpha_2$ is contained in A_j, i.e. meets or traverses no element $\alpha \notin A_j$.

Proof: Assume there is a path P from α_1 to α_2 which contains $\beta_o \notin A_j$. Put $x \equiv 0$

$$d_i(\alpha,\beta) := \begin{cases} 1 & \text{if } (\alpha,\beta) \in R_i \text{ is traversed by P and } h(\alpha) \neq h(\beta) \\ 0 & \text{if } (\alpha,\beta) \in R_i \text{ is traversed by P and } h(\alpha) = h(\beta) \\ -c & \text{otherwise} \end{cases}$$

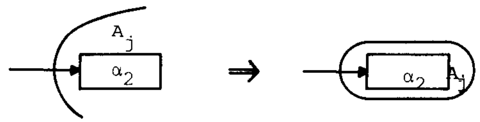
where c denotes the number of sequences (α,β) traversed by P such that $h(\alpha) \neq h(\beta)$. (x,d) is compatible, as any circuit which contains a sequence of positive duration must contain a sequence of duration $-c$ as well. Thus we obtain $\lambda_{\mathcal{R}}(x,d) = \sigma_P(x,d) = c > 0$. In $\mathcal{R}'$ the image of P is a circuit of positive length because of

$h(\alpha_1) = h(\alpha_2)$ and (3.6), which is in contradiction with R3.



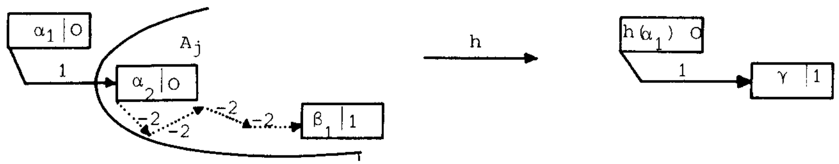
┘

(3.9) If there exists a begin sequence or a normal sequence (α_1,α_2) from $\alpha_1 \notin A_j$ to $\alpha_2 \in A_j$, then A_j is a singleton.



Proof: Let $(\alpha_1,\alpha_2) \in R_{i_o}$, $i_o \in \{b,n\}$. Assume there exists $\beta_1 \in A_j \setminus \{\alpha_2\}$. Put

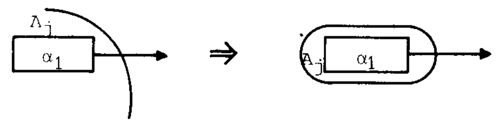
$$x(\alpha) := \begin{cases} 1 & \text{if } \alpha = \beta_1 \\ 0 & \text{otherwise} \end{cases}, \quad d_i(\alpha,\beta) := \begin{cases} 1 & \text{if } (\alpha,\beta) = (\alpha_1,\alpha_2) \wedge i = i_o \\ -2 & \text{otherwise} \end{cases}$$



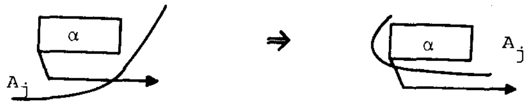
(x,d) is compatible and $\lambda_{\mathcal{R}}(x,d) = 1$, as any path traversing (α_1,α_2) and β_1 must contain a sequence of duration -2. For $\lambda_{\mathcal{R}'}(x',d')$ we obtain 2, which contradicts R4. ┘

Similarly, we obtain:

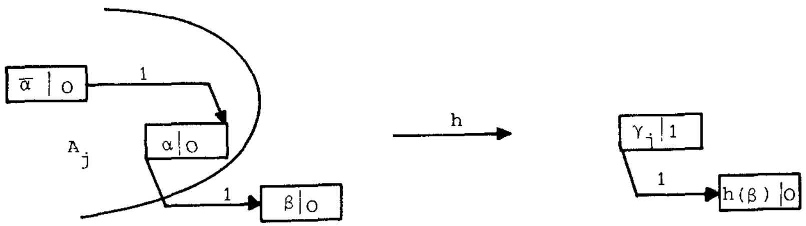
(3.10) If there exists an end sequence or a normal sequence (α_1,α_2) from $\alpha_1 \in A_j$ to $\alpha_2 \notin A_j$, then A_j is a singleton.



(3.11) If there exists a begin sequence or a jump sequence (α,β) from $\alpha \in A_j$ to $\beta \notin A_j$, then α has no predecessors in A_j.



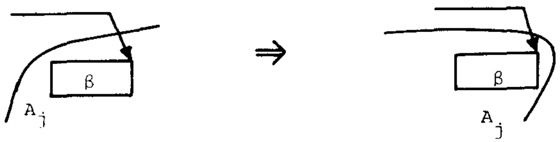
Proof: Let $(\alpha,\beta) \in R_{i_o}$, $i_o \in \{b,j\}$. Suppose that $\bar{\alpha} \in A_j$ is a predecessor of α, i.e. $(\bar{\alpha},\alpha) \in R_{i_1}$ for some $i_1 \in I$. Let the duration of $(\alpha,\beta) \in R_{i_o}$ and $(\bar{\alpha},\alpha) \in R_{i_1}$ be 1,−2 for all other sequences and 0 for all activities. Obviously, (x,d) is compatible and $\lambda_{\mathcal{R}}(x,d) = 2$. In the image network, $\lambda_{\mathcal{R}'}(x',d') = 1$, as there is no path from $h(\beta)$ to $h(\alpha)$ because of (3.8). This is in contradiction with R4.



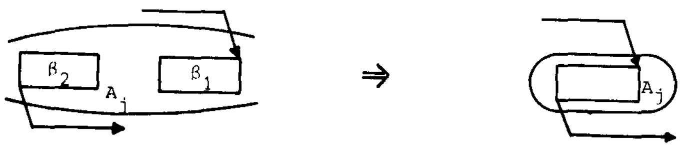
┘

In the same way, we obtain the dual result.

(3.12) If there exists an end sequence or a jump sequence (α,β) from $\alpha \notin A_j$ to $\beta \in A_j$, then β has no successors in A_j.



(3.13) If there exists an end sequence or a jump sequence (α_1,β_1) from $\alpha_1 \notin A_j$ to $\beta_1 \in A_j$ and a begin sequence or a jump sequence (β_2,α_2) from $\beta_2 \in A_j$ to $\alpha_2 \notin A_j$, then A_j is a singleton.



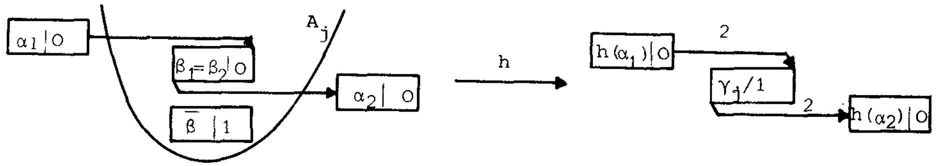
Proof: Assume that $|A_j| \geq 2$. Then the following two cases can occur.

a) $\beta_1 = \beta_2$: Then there exists $\bar{\beta} \in A_j$ such that $\bar{\beta} \neq \beta_1$

$$\text{Put } x(\alpha) := \begin{cases} 1 & \text{if } \alpha = \bar{\beta} \\ 0 & \text{otherwise} \end{cases}$$

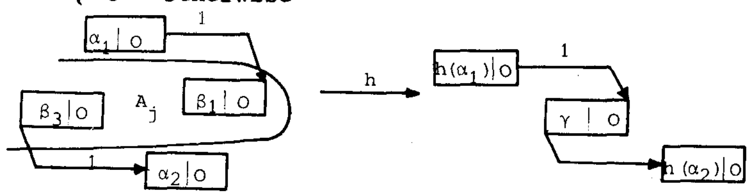
$$d_i(\alpha,\beta) := \begin{cases} 2 & \text{if } (\alpha,\beta) = (\alpha_1,\beta_1) \wedge i \in \{e,j\} \\ & \text{or } (\alpha,\beta) = (\beta_2,\alpha_2) \wedge i \in \{b,j\} \\ -5 & \text{otherwise} \end{cases}$$

(x,d) is compatible and $\lambda_{\mathcal{R}}(x,d) = 4$, as any path which contains $\bar{\beta}$ must contain a sequence of duration −5 as well.
In $\mathcal{R}'$ we obtain $\lambda_{\mathcal{R}'}(x',d') = 3$; a contradiction to R4.



b) $\beta_1 \neq \beta_2$: Because of (3.11),(3.12) and (3.9) there is no path from β_1 to β_2 nor from β_2 to β_1. Put $x \equiv 0$,

$$d_i(\alpha,\beta) := \begin{cases} 1 & \text{if } (\alpha,\beta) = (\alpha_1,\beta_1) \wedge i \in \{e,j\} \\ & \text{or } (\alpha,\beta) = (\beta_2,\alpha_2) \wedge i \in \{b,j\} \\ -3 & \text{otherwise} \end{cases}$$



It follows that (x,d) is compatible, $\lambda_{\mathcal{R}}(x,d) = 1$, but $\lambda_{\mathcal{R}'}(x',d') = 2$.

┘

These results show that partitions which solve the general reduction problem (3.2) have to fulfil very strong conditions. It follows, for example, from (3.8) - (3.13) that

a) each class A_j traversed by a path is a singleton;

b) each sequence (α,β) leaving a non-singleton class A_j (i.e. $\alpha \in A_j$, $\beta \notin A_j$) is a begin sequence or a jump sequence, and has no predecessors in A.
(In the dual case that $\alpha \notin A_j$, $\beta \in A_j$, $|A_j| \geq 2$ the dual statement holds: (α,β) is an end sequence or a jump sequence, and β

has no successors in A.)

Up to now, we have no transmitting rules for image sequences for which all re-image sequences have non-positive durations (see (3.6)). This case may be developed easily:

(3.14) Let $c := \max \{d_i(\alpha,\beta) \mid (\alpha,\beta) \in R_i \cap A_j \times A_k\} \leq 0$. $d_i'(\gamma_j,\gamma_k)$ may be chosen arbitrarily ≤ 0 if a) or b) holds, or arbitrarily $\leq c$ if c) or d) holds.

a) $(\gamma_j,\gamma_k) \in R_b' \cup R_j'$ and γ_j has no predecessors

b) $(\gamma_j,\gamma_k) \in R_e' \cup R_j'$ and γ_j has no successors

c) $(\gamma_j,\gamma_k) \in R_b' \cup R_j'$ and γ_k is the only predecessor of γ_j

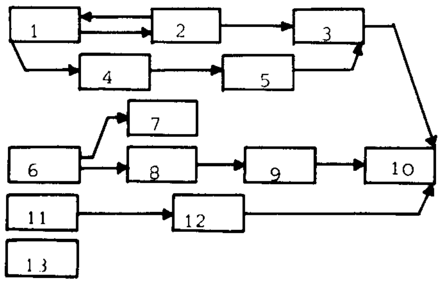
d) $(\gamma_j,\gamma_k) \in R_e' \cup R_j'$ and γ_j is the only successor of γ_k.

In all other cases, $d_i'(\gamma_j,\gamma_k) = c$.

Because of the very restrictive form of the partitions π it follows immediately that the conditions (3.4) - (3.14) are also sufficient for the solution of (3.2).

THEOREM 3: *The solutions of the general reduction problem (3.2) are exactly given by all partitions π which fulfil (3.8) - (3.13), with the image structure and the transmitting rules being as given by (3.7) and (3.4), (3.6) and (3.14) respectively.*

(3.15) EXAMPLE: Let $\mathcal{R}$ be given by the following diagram



$\pi_1 = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6,7,8,9,10\},\{11\},\{12\},\{13\}\}$ and

$\pi_2 = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6,7,8,9,10,13\},\{11\},\{12\}\}$

are solutions of the general reduction problem, whereas

$\pi_3 = \{\{1,2,3,4,5\},\{6,7,8,9,10\},\{11\},\{12\},\{13\}\}$ is not, since $A_j = \{1,2,3,4,5\}$ violates (3.10).

4. THE RESTRICTED REDUCTION PROBLEM

The solutions of the general reduction problem are, of course, unsuited for practical application. As the limited form of solution mainly depends on negative sequence durations being allowed, the question narurally arises of whether the range of possibilities can be increased by excluding negative durations and circuits. [16)]

(4.1) THE LIMITED REDUCTION PROBLEM

Let $\mathcal{R} = (A,(R_i)_{i\in I})$ be a network structure without circuits. Characterize all partitions $\pi = \{A_j \mid j \in J\}$ of A (together with their assigned image structure $\mathcal{R}'$ and transmitting rules according to (3.1)) which fulfil the following conditions:

R1' = R1 CONSERVATION OF STRUCTURE

R2' = R2 LOCAL TRANSMISSION

R3' For each non-negative duration function (x,d) on $\mathcal{R}$ (x',d') is non-negative and compatible on $\mathcal{R}'$

COMPATIBLE AND NON-NEGATIVE TRANSMISSION

R4' For each non-negative duration function (x,d) on $\mathcal{R}$ we have $\lambda_{\mathcal{R}}(x,d) = \lambda_{\mathcal{R}'}(x',d')$

CONSERVATION OF THE SHORTEST OVERALL DURATION

From the considerations of §3, we obtain the following theorem.

16) Another approach involves weakening the local transmission condition R2 for sequences as follows.
Besides the variables $d_i(\alpha,\beta)$, $(\alpha,\beta) \in R_i \cap A_j \times A_k$, $d_i'(\gamma_j,\gamma_k)$ may also depend on the durations of the activities and sequences contained in $A_j \cup A_k$. This approach leads to complicated transmitting rules, however, which may not even be unique. Furthermore, the classes A_j are no longer fully characterized by the sequences entering or leaving them. For these reasons, this approach will not be treated here any further.

THEOREM 4: *If $\pi = \{A_j \mid j \in J\}$ is a solution of the restricted reduction problem (4.1), the assigned image structure $\mathcal{R}'$ and the transmitting rules must fulfil the following conditions:*

a) *For all $\alpha, \beta \in A$ such that $h(\alpha) \neq h(\beta)$ and each $i \in I$, we have $(\alpha,\beta) \in R_i \leftrightarrow (h(\alpha),h(\beta)) \in R'_i$*

b) *$\mathcal{R}'$ contains no circuits*

c) *$x'(\gamma_j) = \lambda_{\mathcal{R}|A_j}(x,d)$ for all $\gamma_j \in A'$*

d) *$d_i(\gamma_j,\gamma_k) = \max\ \{d_i(\alpha,\beta) \mid (\alpha,\beta) \in R_i \cap A_j \times A_k\}$ for all $(\gamma_j,\gamma_k) \in R'_i$.*

Proof: Statements (3.3),(3.4),(3.5) and (3.7) (but not (3.6)) remain true, this being shown in the same way as in §3.[17]
This proves a) and c).
Now assume that C is a circuit in $\mathcal{R}'$ which traverses the sequence $(\gamma_j,\gamma_k) \in R'_{i_o}$. Put

$$x \equiv 0, \quad d_i(\alpha,\beta) := \begin{cases} 1 & (\alpha,\beta) \in R_{i_o} \cap A_j \times A_k \wedge i = i_o \\ 0 & \text{otherwise} \end{cases}$$

Then, $\lambda_{\mathcal{R}'}(x',d') \overset{R4'}{=} \lambda_{\mathcal{R}}(x,d) \geq 1$ and, because of (3.3),(3.4) $d'_i \equiv 0$ except for $d'_{i_o}(\gamma_j,\gamma_k)$. Therefore $\lambda_{\mathcal{R}'}(x',d') = d'_{i_o}(\gamma_j,\gamma_k) \geq 1$, which means that C is a circuit of positive length in $(\mathcal{R}',x',d')$. This contradicts R3' and thus proves b).
In order to show d), we need an analogue of (3.8):

(4.2) Each path P from $\alpha_1 \in A_j$ to $\alpha_2 \in A_j$, $\alpha_1 \neq \alpha_2$ is contained in A_j, i.e. meets or traverses no element $\alpha \notin A_j$.

Proof: Assume that P contains $\alpha \notin A_j$. Because of a) the image of P is a circuit in $\mathcal{R}'$, and this contradicts b).

Now the rest of Theorem 4 is proved as follows:

Let $(x,d) \geq 0$ and $(\gamma_j,\gamma_k) \in R'_{i_o}$. Put $y \equiv 0$ and

$$l_i(\alpha,\beta) := \begin{cases} d_i(\alpha,\beta) & \text{if } (\alpha,\beta) \in R_{i_o} \cap A_j \times A_k \wedge i = i_o \\ 0 & \text{otherwise} \end{cases}$$

(4.2) implies that there is no path traversing more than one sequence $(\alpha,\beta) \in R_{i_o} \cap A_j \times A_k$. Therefore,

17) Of course, " < 0 " must be replaced by " = 0 ", etc.

$$\max\ \{d_{i_o}(\alpha,\beta) \mid (\alpha,\beta) \in R_{i_o} \cap A_j \times A_k\}$$
$$= \max\ \{l_{i_o}(\alpha,\beta) \mid (\alpha,\beta) \in R_{i_o} \cap A_j \times A_k\}$$
$$\overset{(4.2)}{=} \lambda_{\mathcal{R}}(y,l) \overset{R4'}{=} \lambda_{\mathcal{R}'}(y,l) \overset{(3.3),(3.4),R2'}{=} d'_{i_o}(\gamma_j,\gamma_k).$$
┘

For the classes A_j, (3.11),(3.12) and (3.13) remain valid, this being proved with the same arguments as in §3.

The main (and essential!) difference is caused by the fact that (3.9) and (3.10) are replaced by weaker conditions.

THEOREM 5: *Let $\pi = \{A_j \mid j \in J\}$ be a solution of the restricted reduction problem (4.1). For the classes A_j of π, the following conditions hold:*

a) *Each path from $\alpha_1 \in A_j$ to $\alpha_2 \in A_j$ is contained in A_j*

b) *If there exists a begin sequence or a normal sequence from $\alpha_1 \notin A_j$ to $\alpha_2 \in A_j$, then α_2 is connected with each $\alpha \in A_j \setminus \{\alpha_2\}$ by a path $P(\alpha_2,\alpha)$ which traverses α positively.*

c) *If there exists an end sequence or a normal sequence (α_1,α_2) from $\alpha_1 \in A_j$ to $\alpha_2 \notin A_j$, then each $\alpha \in A_j \setminus \{\alpha_1\}$ is connected with α_1 by a path $P(\alpha,\alpha_1)$ which traverses α positively.*

d) *If there exists an end sequence or a jump sequence (α_1,α_2) from $\alpha_1 \notin A_j$ to $\alpha_2 \in A_j$, then α_2 has no successors in A_j.*

e) *If there exists a begin sequence or a jump sequence (α_1,α_2) from $\alpha_1 \in A_j$ to $\alpha_2 \notin A_j$, then α_1 has no predecessors in A_j.*

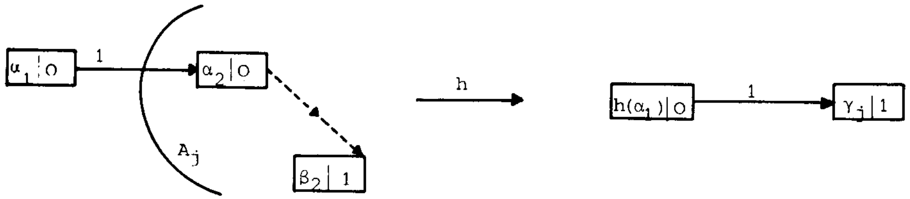
f) *If there exists an end sequence or a jump sequence (α_1,β_1) from $\alpha_1 \notin A_j$ to $\beta_1 \in A_j$ and a begin sequence or a jump sequence (β_2,α_2) from $\beta_2 \in A_j$ to $\alpha_2 \notin A_j$, then A_j is a singleton.*

Proof: a) is proved by (4.2)
Reformulation of the proofs of (3.11) - (3.13), replacing durations " < 0 " by durations " = 0 " and use of the fact that $\mathcal{R}$ and $\mathcal{R}'$ (Theorem 4) contain no circuits yields d), e) and f).

b) Let $(\alpha_1,\alpha_2) \in R_{i_o}$, $i_o \in \{b,n\}$. Assume that there exists $\beta_2 \in A_j$ such that there is no path $P(\alpha_2,\beta_2)$ which traverses β_2.

Put $x(\alpha) := \begin{cases} 1 & \text{if } \alpha = \beta_2 \\ 0 & \text{otherwise} \end{cases}$

$d_i(\alpha,\beta) := \begin{cases} 1 & \text{if } (\alpha,\beta) = (\alpha_1,\alpha_2) \wedge i = i_o \\ 0 & \text{otherwise} \end{cases}$



Because of (4.2) there is no path from β_2 to α_1 in $\mathcal{R}$, so $\lambda_{\mathcal{R}}(x,d) = 1 \neq \lambda_{\mathcal{R}'}(x',d') = 2$, which contradicts R4'.

c) is proved in the same way. ┘

DEFINITION: Let $\mathcal{R} = (A,(R_i)_{i\in I})$ be a network structure without circuits. A partition π of A is said to be *reducing* if it fulfils the conclusions of Theorem 5.

Theorem 5 states that solutions of the restricted reduction problem must be reducing. The sufficiency of this condition is shown by the following theorem.

THEOREM 6: *The solutions of the restricted reduction problem (4.1) are exactly given by all reducing partitions of the given network structure, the image structure and the transmitting rules being given by Theorem 4,a),c),d).*

Proof: It remains to show that a partition for which the assumptions of Theorem 6 hold fulfils condition R4' of (4.1).
To show this, let $N = (\mathcal{R},x,d)$ be a network without circuits, $(x,d) > 0$, and π be a reducing partition of A. Obviously, the one-step transition from $\mathcal{R}$ to the image network $\mathcal{R}'$ can be replaced by a step-by-step transition, where only one non-singleton class A_j of π is mapped onto its corresponding image activity γ_j and all activities $\alpha \notin A_j$ remain unchanged. [18)]

18) If $|A_j| \geq 2$ $\pi_j := \{A_j, \{\alpha\} \mid \alpha \in A \setminus A_j\}$ is a reducing partition of $\mathcal{R}$ and $\pi_j' := (\pi \setminus \{A_j\}) \cup \{\gamma_j\}$ is a reducing partition of the image structure of $\mathcal{R}$ induced by π.

We can therefore assume w.l.o.g. that π contains only one non-singleton class, i.e. $\pi = \{B,\{\alpha\} \mid \alpha \in A \setminus B\}$, $|B| \geq 2$.

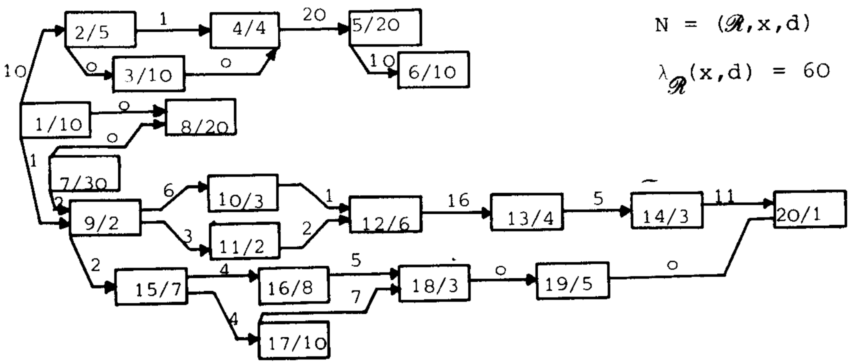
Let $h(\alpha) := \begin{cases} \alpha & \text{if } \alpha \notin B \\ \gamma & \text{otherwise} \end{cases}$ be the canonical mapping assigned to π.

It follows from Theorem 5 that the set $\mathbf{P}'$ of all paths in the image structure $\mathcal{R}'$ is equal to the set of all images of paths in $\mathcal{R}$, i.e. $\mathbf{P}' = \{h(P) \mid P \in \mathbf{P}\}$ where $\mathbf{P}$ denotes the set of all paths in $\mathcal{R}$.

a) Let P_o be a critical path in N, i.e. $\lambda_{\mathcal{R}}(x,d) = \sigma_{P_o}(x,d)$. If P_o traverses no activity from B positively, it contains at most one activity from B, since π is reducing. Then:
$\lambda_{\mathcal{R}}(x,d) = \sigma_{P_o}(x,d) \overset{\text{Th.4}}{=} \sigma_{h(P_o)}(x',d') \leq \lambda_{\mathcal{R}'}(x',d')$, as $h(P_o) \in \mathbf{P}'$
If P_o traverses an activity from B positively, the part $P_o \cap B$ of P_o which is contained in B, is critical in $N|B$, i.e. $\sigma_{P_o \cap B}(x,d) = \lambda_{\mathcal{R}|B}(x,d) = x'(\gamma)$. Again we obtain
$\lambda_{\mathcal{R}}(x,d) = \sigma_{P_o}(x,d) \overset{\text{Th.4}}{=} \sigma_{h(P_o)}(x',d') \leq \lambda_{\mathcal{R}'}(x',d')$, as $h(P_o) \in \mathbf{P}'$.

b) Let P' be critical in N'. Because of Theorem 4,c),d) and $\mathbf{P}' = \{h(P) \mid P \in \mathbf{P}\}$ there exists a path P_o in N such that $\sigma_{P_o}(x,d) = \sigma_{P_o'}(x',d')$
Then: $\lambda_{\mathcal{R}'}(x',d') = \sigma_{P_o'}(x',d') = \sigma_{P_o}(x,d) \leq \lambda_{\mathcal{R}}(x,d)$.
From a) and b) we obtain $\lambda_{\mathcal{R}'}(x',d') = \lambda_{\mathcal{R}}(x,d)$, which proves the theorem. ┘

(4.3) EXAMPLE: Let N be given by following diagram



$N = (\mathcal{R},x,d)$

$\lambda_{\mathcal{R}}(x,d) = 60$

$$\pi_1 = \{\overbrace{\{5,6\}}^{A_2},\overbrace{\{1,7,8\}}^{A_3},\{2\},\{3\},\{4\},\overbrace{\{9,10,11,12\}}^{A_1},\overbrace{\{13,14\}}^{A_4},\overbrace{\{18,19\}}^{A_5},\{15\},\{16\},\{17\},\{20\}\}$$

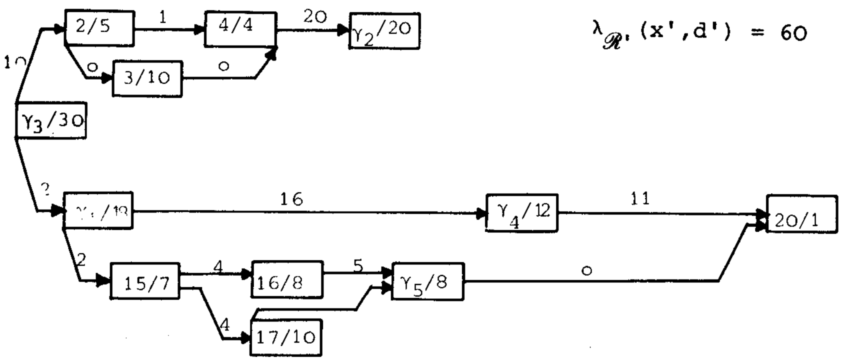
$$\pi_2 = \{\{2,3,4,5,6\},\{1,7,8\},\{9,10,11,12,13,14\},\{15\},\{16\},\{17\},\{18\},\{19\},\{20\}\}$$

$$\pi_3 = \{\{2,3,4,5,6\},\{1,7,8\},\{9,10,11,12,13,14,15,16,17,18,19,20\}\}$$

are reducing partitions of N.

{2,4} and {15,16,17,18} are not classes of a reducing partition, as parts f) and c) of Theorem 5 are then violated.

The image network $\mathcal{R}'$ assigned to π_1 is given by the following diagram.



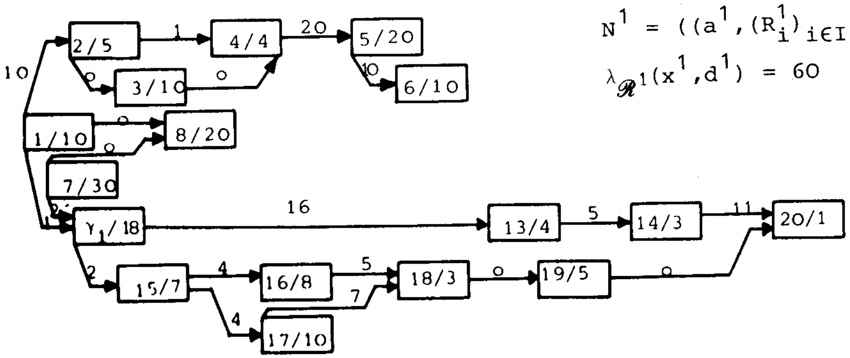
$\lambda_{\mathcal{R}'}(x',d') = 60$

Finally, we give a step-by-step transition from $\mathcal{R}$ to $\mathcal{R}'$ as used in the proof of Theorem 6:

Step 1: reduction of A_1

$N^1 = ((a^1,(R_i^1)_{i\in I}),x^1,d^1)$

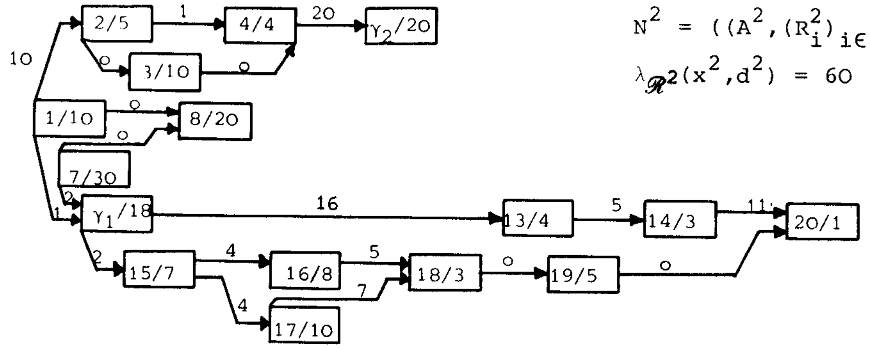
$\lambda_{\mathcal{R}^1}(x^1,d^1) = 60$



Step 2: reduction of A_2

$N^2 = ((A^2,(R_i^2)_{i\in I})x^2,d^2)$

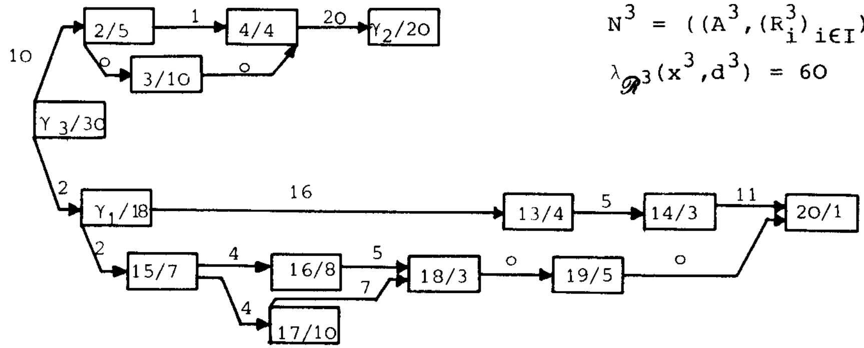
$\lambda_{\mathcal{R}^2}(x^2,d^2) = 60$



Step 3: reduction of A_3

$N^3 = ((A^3,(R_i^3)_{i\in I}),x^3,d^3)$

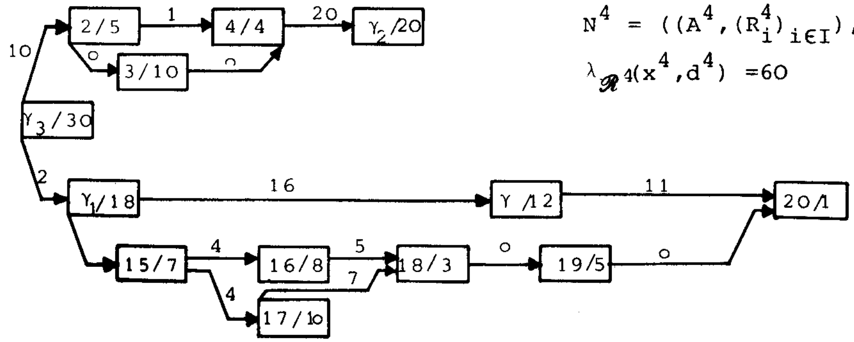
$\lambda_{\mathcal{R}^3}(x^3,d^3) = 60$



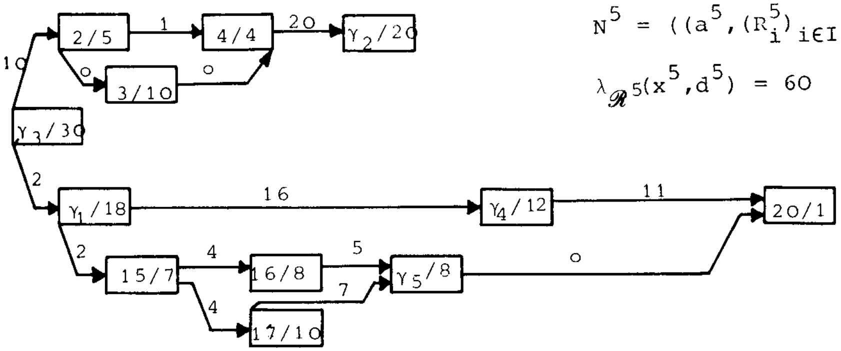
Step 4: reduction of A_4

$N^4 = ((A^4,(R_i^4)_{i\in I}),x^4,d^4)$

$\lambda_{\mathcal{R}^4}(x^4,d^4) = 60$



Step 5: reduction of A_5



$N^5 = ((a^5,(R_i^5)_{i\in I}),x^5,d^5)$

$\lambda_{\mathscr{R}^5}(x^5,d^5) = 60$

In comparison with the general reduction problem (3.2), the situation has improved. The conditions for the partitions are weaker, which of course, increases the possibilities of reduction. Furthermore, the classes of a reducing partition are fully characterized by local properties, namely the behaviour of the sequences which enter or leave them.

This fact of course facilitates the search for reducing partitions enormously. (cf. Footnote 16)

In addition, the second aim of the reduction theory, reduced computation of characteristic activity times and (extension) floats, can be achieved. This will be shown in the last paragraph.

Comparison with the situation in CPM networks (cf. [8]), however, makes the reduction possibilities for the restricted reduction problem seem almost trivial and without great scope for practical application.

Summing up, it can be said that, compared with CPM networks, the generalization of the network structure must be paid for with a considerable limitation of reduction possibilities.

5. REDUCTION THEOREMS FOR CHARACTERISTIC ACTIVITY TIMES AND FLOATS

The solutions of the restricted reduction problem enable two-step computation of the times ES, LF and ES^I, LF^T. From them, all characteristic times and floats can be derived using Definition 4 and 5.

Furthermore, for certain classes of activities we obtain reduction formulas for floats and characteristic times that are in simple form, and which yield new information about the nature of some floats.

The methods used to prove the statements of this paragraph are similar to that used in the proof of Theorem 6.

We assume w.l.o.g. that the given reducing partition of N has only one non-singleton class, and consider the set of paths which defines the float or characteristic time at question in N and its image N'. For this reason, the proofs shall be omitted here.

For the rest of this chapter, we make the following assumptions: Let $N = (\mathscr{R},x,d)$ be a network with a network structure $\mathscr{R} = (A,(R_i)_{i\in I})$ without circuits and a non-negative duration function (x,d). Let π be a reducing partition of A and, for $\alpha\in A$, let A_α denote the class of π which contains α. Let h denote the canonical mapping assigned to π and $N' = (\mathscr{R}',x',d')$ the image network. For each $\alpha\in A$, let $P^I(\alpha) := \{\beta\in A \mid (\beta,\alpha)\in R_s \cup R_n\}$, $P^T(\alpha) := \{\beta\in A \mid (\beta,\alpha)\in R_e \cup R_j\}$ and $P(\alpha) := P^I(\alpha)\cup P^T(\alpha)$ denote the set of the predecessors of α at its initial point, at its terminal point, and of all its predecessors, respectively. Dually, define
$S^I(\alpha) := \{\beta\in A \mid (\alpha,\beta)\in R_s \cup R_j\}$,
$S^T(\alpha) := \{\beta\in A \mid (\alpha,\beta)\in R_e \cup R_n\}$, $S(\alpha) := S^I(\alpha)\cup S^T(\alpha)$.

$I := \{\alpha\in A \mid P(\alpha) = \emptyset\}$ is called the set of *initial activities of* A.

$T := \{\alpha\in A \mid S(\alpha) = \emptyset\}$ is called the set of *terminal activities of* A.

Similarly,

$I_\alpha := \{\beta\in A_\alpha \mid P(\beta)\cap A_\alpha = \emptyset\}$ is called the set of *initial activities of* A_α.

$T_\alpha := \{\beta\in A_\alpha \mid S(\beta)\cap A_\alpha = \emptyset\}$ is called the set of *terminal activities of* A_α.

THEOREM 7:[19] REDUCTION-FORMULAS FOR ES AND LF:

a) *(i)* $ES_N(\alpha) = ES_{N'}(h(\alpha))$ *if* $|A_\alpha| = 1$

(ii) $ES_N(\alpha) = ES^I_{N'}(h(\alpha)) + ES_{N|A}(\alpha)$ *if* $P^T(\alpha)\setminus A_\alpha = \emptyset$

19) The cases (i) - (iv) do not exclude each other. They are chosen in order to give as simple as possible expressions for ES and LF.

$$(iii)\quad ES_N(\alpha) = \max \begin{cases} ES^{I}_{N'}(h(\alpha)) + ES_{N|A_\alpha}(\alpha) \\ \max\limits^{20)}_{(\beta,\alpha)\in R_e \wedge \beta\notin A_\alpha} [ES_{N'}(h(\beta)) + x'(h(\beta)) + d_e(\beta,\alpha) - x(\alpha)] \\ \max\limits_{(\beta,\alpha)\in R_j \wedge \beta\notin A_\alpha} [ES_{N'}(h(\beta)) + d_j(\beta,\alpha) - x(\alpha)] \end{cases}$$

$\text{if } P^T(\alpha) \setminus A \neq \emptyset$

$(iv)\quad ES_N(\alpha) = ES_{N'}(h(\alpha)) - x(\alpha) + x'(h(\alpha)) \qquad \text{if } T_\alpha = \{\alpha\}$

b) $(i)\quad LF_N(\alpha) = LF_{N'}(h(\alpha)) \qquad \text{if } |A_\alpha| = 1$

$(ii)\quad LF_N(\alpha) = LF^{T}_{N'}(h(\alpha)) - \lambda_{\mathscr{R}|A_\alpha}(x,d) + LF_{N|A_\alpha}(\alpha) \qquad \text{if } S^I(\alpha) \setminus A_\alpha = \emptyset$

$$(iii)\quad LF_N(\alpha) = \min \begin{cases} LF^{T}_{N'}(h(\alpha)) - \lambda_{\mathscr{R}|A}(x,d) + LF_{N|A_\alpha}(\alpha) \\ \min\limits^{21)}_{(\alpha,\beta)\in R_s \wedge \beta\notin A_\alpha} [LF_{N'}(h(\beta)) - x'(h(\beta)) - d_s(\alpha,\beta) + x(\alpha)] \\ \min\limits_{(\alpha,\beta)\in R_j \wedge \beta\notin A_\alpha} [LF_{N'}(h(\beta)) - d_j(\alpha,\beta) + x(\alpha)] \end{cases}$$

$\text{if } S^I(\alpha) \setminus A_\alpha = \emptyset$

$(iv)\quad LF_N(\alpha) = LF_{N'}(h(\alpha)) + x(\alpha) - x'(h(\alpha)) \qquad \text{if } I_\alpha = \{\alpha\}$

The following reduction theorems for the other characteristic activities and floats cover only those activities for which a short and significant reduction formula can be given.

First, we introduce the following abbreviations:

ES 1: $\longleftrightarrow\quad P(\alpha) = P(h(\alpha)) = \emptyset \vee |A_\alpha| = 1$

ES 2: $\longleftrightarrow\quad P^T(\alpha) \setminus A = \emptyset$

LF 1: $\longleftrightarrow\quad S(\alpha) = S(h(\alpha)) = \emptyset \vee |A_\alpha| = 1$

LF 2: $\longleftrightarrow\quad S^I(\alpha) \setminus A = \emptyset$

---

20) The maximum over the void set is defined to be zero.

21) The minimum over the void set is defined to be $\lambda_{\mathscr{R}}(x,d)$.

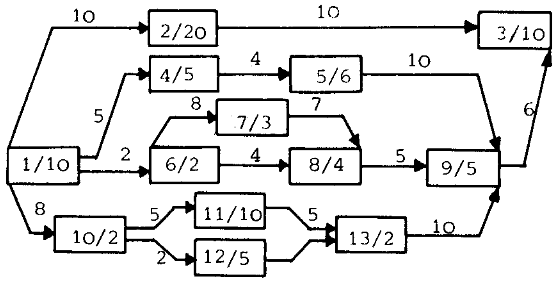
EI 1: $\longleftrightarrow\quad S(\alpha) = S(h(\alpha)) = \emptyset \vee [|A_\alpha| = 1 \wedge \forall \beta \in S(\alpha): |A_\beta| = 1]$

EI 2: $\longleftrightarrow\quad S(\alpha) \neq \emptyset \wedge S(\alpha) \subset A_\alpha \wedge \forall \beta \in S(\alpha): P^T(\beta) \setminus A_\beta = \emptyset$

LI 1: $\longleftrightarrow\quad P(\alpha) = P(h(\alpha)) = \emptyset \vee [|A_\alpha| = 1 \wedge \forall \beta \in P(\alpha): |A_\beta| = 1]$

LI 2: $\longleftrightarrow\quad P(\alpha) \neq \emptyset \wedge P(\alpha) \subset A_\alpha \wedge \forall \beta \in P(\alpha): S^I(\beta) \setminus A_\beta = \emptyset$

(5.1) EXAMPLE: Let N be given by the following diagram



$\pi = \{\{1\},\{2\},\{3\},\{4,5\},\{6,7,8,9\},\{10,11,12,13\}\}$ is a reducing partition. $\alpha = 1$ fulfils the following conditions: ES 1, ES 2, LF 1 and LI 1, $\alpha = 10$ fulfils ES 2, LF 2 and EI 2.

THEOREM 8: SPECIAL REDUCTION FORMULAS

a) $EI_N(\alpha) = EI_{N'}(h(\alpha))$ *if α fulfils EI 1*

$EI_N(\alpha) = ES^{I}_{N'}(h(\alpha)) + EI_{N|A_\alpha}(\alpha)$ *if α fulfils EI 2*

$LI_N(\alpha) = LI_{N'}(h(\alpha))$ *if α fulfils LI 1*

$LI_N(\alpha) = LF^{T}_{N'}(h(\alpha)) - x'(h(\alpha)) + LI_{N|A_\alpha}(\alpha)$ *if α fulfils LI 2*

b) $TF_N(\alpha) = TF_{N'}(h(\alpha))$ *if α fulfils ES 1 and LF 1*

$TF_N(\alpha) = TF_{N'}(h(\alpha)) + TE^{T}_{N'}(h(\alpha)) + TE^{I}_{N'}(h(\alpha)) + TF_{N|A_\alpha}(\alpha)$ *if α fulfils ES 2 and LF 2*

$FF_N(\alpha) = FF_{N'}(h(\alpha))$ *if α fulfils EI 1 and ES 1*

$FF_N(\alpha) = FF_{N|A_\alpha}(\alpha)$ *if α fulfils EI 2 and ES 2*

$IF_N(\alpha) = IF_{N'}(h(\alpha))$ *if α fulfils LI 1 and EI 1*

$IF_N(\alpha) = IF_{N|A_\alpha}(\alpha) - TF_{N'}(h(\alpha)) - TE^T_{N'}(h(\alpha)) - TE^I_{N'}(h(\alpha))$ *if* α *fulfils LI 2 and EI 2*

c) $TE^I_N(\alpha) = TE^I_{N'}(h(\alpha))$ *if* α *fulfils ES 1*

$TE^I_N(\alpha) = TE^I_{N|A_\alpha}(\alpha)$ *if* α *fulfils ES 2*

$TE^T_N(\alpha) = TE^T_{N'}(h(\alpha))$ *if* α *fulfils LF 1*

$TE^T_N(\alpha) = TE^T_{N|A_\alpha}(\alpha)$ *if* α *fulfils LF 2*

$IE^I_N(\alpha) = IE^I_{N'}(h(\alpha))$ *if* α *fulfils LI 1*

$IE^I_N(\alpha) = IE^I_{N|A_\alpha}(\alpha)$ *if* α *fulfils LI 2*

$IE^T_N(\alpha) = IE^T_{N'}(h(\alpha))$ *if* α *fulfils EI 1*

$IE^T_N(\alpha) = IE^T_{N|A_\alpha}(\alpha)$ *if* α *fulfils EI 2*

Although reduction formulas have not been given for all $\alpha \in A$, we can derive significant properties with regard to the nature of several floats.

REMARK

a) TF and IF are *global* floats, i.e. to compute them we need information from the total network.

b) The FF and all extension floats are *local* floats, i.e. to compute them for α we require only information from the smallest class A_j of a reducing partition which contains α such that ES 2, LF 2, EI 2 and LI 2 hold.

We conclude this paragraph with an example:

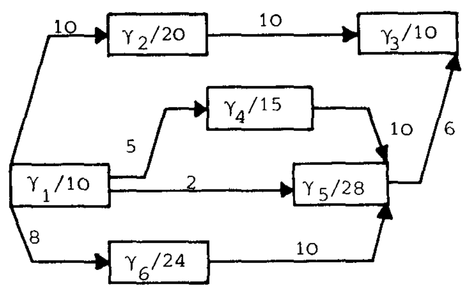
(5.2) EXAMPLE (CONTINUATION OF (5.1))

Let N and π be given as in (5.1).

For N, we obtain the following times and floats:

activity α	$x(\alpha)$	ES	LF	EI	LI	TF	FF	IF	TE^I	TE^T	IE^I	IE^T
1	10	0	10	10	0	0	0	0	0	4	0	0
2	20	10	30	30	10	0	0	0	0	0	0	0
3	10	40	50	50	40	0	0	0	0	0	0	0
4	5	15	24	20	15	4	0	0	0	0	0	0
5	6	24	34	32	28	4	2	-2	0	0	0	0
6	2	12	18	14	12	4	0	0	0	8	0	8
7	3	20	27	23	24	4	0	-4	0	0	0	0
8	4	26	34	32	30	4	2	-2	8	0	8	0
9	5	37	44	44	39	2	2	0	2	0	0	0
10	2	8	12	10	8	2	0	0	0	0	0	0
11	10	15	27	25	17	2	0	-2	0	0	0	0
12	5	12	28	26	14	11	9	7	0	0	0	0
13	2	30	34	32	32	2	0	-2	0	0	0	0

The image network $N' = (\mathscr{R}', x')$ is given by the following diagram:



γ_j corresponds to the class A_j of π, where

$A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3\}$,

$A_4 = \{4,5\}$, $A_5 = \{6,7,8,9\}$,

$A_6 = \{10,11,12,13\}$.

Times and floats of N' and the sub-networks $N|A_j$, j = 5,6 are given by the following tables:

1. Times/floats of N'

activity γ	$x(\gamma)$	ES	LF	EI	LI	TF	FF	IF	TE^I	TE^T	IE^I	IE^T
γ_1	10	0	10	10	0	0	0	0	0	4	0	0
γ_2	20	10	30	30	10	0	0	0	0	0	0	0
γ_3	10	40	50	50	40	0	0	0	0	0	0	0
γ_4	15	15	34	32	15	4	2	2	0	0	0	0
γ_5	28	14	44	44	16	2	2	0	2	0	4	0
γ_6	24	8	34	32	8	2	0	0	0	0	0	0

2. Times/floats of $N|A_5$ (i.e. $ES_{N|A_5}$, $LF_{N|A_5}$, etc.)

activity α	$x(\alpha)$	ES	LF	EI	LI	TF	FF	IF	TE^I	TE^T	IE^I	IE^T
6	2	0	2	2	0	0	0	0	0	8	0	8
7	3	8	11	11	8	0	0	0	0	0	0	0
8	4	14	18	18	14	0	0	0	8	0	8	0
9	5	23	28	28	23	0	0	0	0	0	0	0

3. Times/floats of $N|A_6$ (i.e. $ES_{N|A_6}$, $LF_{N|A_6}$, etc.)

activity α	$x(\alpha)$	ES	LF	EI	LI	TF	FF	IF	TE^I	TE^T	IE^I	IE^T
10	2	0	2	2	0	0	0	0	0	0	0	0
11	10	7	17	17	7	0	0	0	0	0	0	0
12	5	4	18	18	4	9	9	9	0	0	0	0
13	2	22	24	24	22	0	0	0	0	0	0	0

Examples of two-step computation:

(i) For $\alpha = 6$ ES 2 and LF 2 hold.
$TF_N(6) = 4$, $h(6) = \gamma_5$
$TF_{N'}(\gamma_5) + TE^T_{N'}(\gamma_5) + TE^I_{N'}(\gamma_5) + TF_{N|A_5}(6) = 2+0+2+0 = 4$

(ii) For $\alpha = 12$ ES 2, LF 2, EI 2 and LI 2 hold. $h(12) = \gamma_6$
$FF_N(12) = FF_{N|A_6}(12) = 9$, $IF_N(12) = 7$
$IF_{N|A_6}(12) - [TF_{N'}(\gamma_6) + TE^T_{N'}(\gamma_6) + TE^I_{N'}(\gamma_6)] = 9-(2+0+0)$
$= 7$

(iii) For $\alpha = 8$ ES 2 holds, while EI 2 does not. $h(8) = \gamma_5$
$TE^I_N(8) = TE^I_{N|A_5}(8)$
$FF_N(8) = 2 \neq 0 = FF_{N|A_5}(8)$ (EI 2 does not hold).

Finally, we give two-step computation of $ES_N(\alpha)$, $LF_N(\alpha)$, $\alpha \in A_6$, as developed in Theorem 7.

activity	ES_N	$ES^I_{N'}$	$ES_{N\|A_6}$	LF_N	$LF^T_{N'}$	$\lambda_{\mathscr{R}\|A_6}(x,d)$	$LF_{N\|A_6}$
10	8	8	0	12	34	24	2
11	15	8	7	27	34	24	17
12	12	8	4	28	34	24	18
13	30	8	22	34	34	24	24

For all $\alpha \in A_6$ we have $ES_N(\alpha) = ES^I_{N'}(\gamma_6) + ES_{N|A_6}$ and
$LF_N(\alpha) = LF^T_{N'}(\gamma_6) - \lambda_{\mathscr{R}|A_6}(x,d) + LF^T_{N|A_6}(\alpha)$.

To compute $ES^I_{N'}$ and $LF^T_{N'}$ the formulas $TE^I_{N'}(\gamma_j) = ES_{N'}(\gamma_j) - ES^I_{N'}(\gamma_j)$ and $TE^T_{N'}(\gamma_j) = LF^T_{N'}(\gamma_j) - LF_{N'}(\gamma_j)$ can be used, cf. definition of TE.

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