Relations

Laura Kovács

Preliminaries – Sets (Menge)

- A set is a group of objects;
- Objects in a set are called elements or members;
- One way of describing a set is by listing its elements inside braces.

  For example: {7,14,21,28}; finite set

  set of natural numbers: $\mathbb{N} = \{0,1,2,\ldots\}$; infinite set

  set of integer numbers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$; infinite set
- The set with 0 elements is the empty set, denoted by Ø.
- Set membership is denoted by the symbol: $\in$.

  For example: $7 \in \{7,14,21,28\}$;
- Set nonmembership is denoted by the symbol: $\notin$.

  For example: $8 \notin \{7,14,21,28\}$.

Preliminaries – Operation on Sets

Given two sets A and B.

- A is a subset (Teilmenge) of B, written $A \subseteq B$, if:
  every member of A is also an element of B.

  Thus, $A \subseteq B$ is logically equivalent to $\forall$ x:: $x \in A \Rightarrow x \in B$.

- A is a proper subset of B, written $A \subsetneq B$, if:
  A is a subset of B and not equal to B.

  Thus, $A \subsetneq B$ is logically equivalent to $A \subseteq B \wedge A \neq B$.

- The union of A and B is the set $A \cup B$ obtained by combining *all* elements of A and B;

- The intersection of A and B is the set $A \cap B$ of elements that are both in A and B;

- The complement of A is the set A of all elements that are not in A;

- The Cartesian product (karthesischen Produkt) of A and B is the set A x B of all pairs (a,b) such that $a \in A$ and $b \in B$.

  One may write: $A x B = \{(a,b) \mid a \in A \text{ and } b \in B\}$

Relations (Relationen)

Given the sets $A_1, A_2, \ldots, A_n$.

- An n-ary relation R is a subset of the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$:

$$R \subseteq A_1 \times A_2 \times \ldots \times A_n$$

- A binary relation R is a subset of the Cartesian product $A_1 \times A_2$:

$$R \subseteq A_1 \times A_2$$

Note: Binary relation is also called a 2-ary relation.

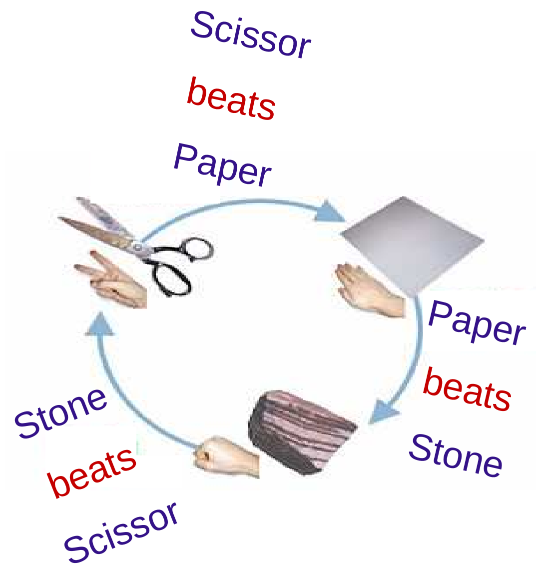
For $(a,b) \in R$ one also writes aRb.

Note: The statement aRb means that aRb is True.

If $A_1 = A_2$ we say that *R is a relation over* A_1.

Examples: <, >, = are binary relations over numbers.

Relations - Example



Scissors-Paper-Stone game:

-Two players simultaneously select a member from the set {Scissor, Paper, Stone};

- If selections are the same, the game starts over;

- If selection differ, one player wins according to the picture.

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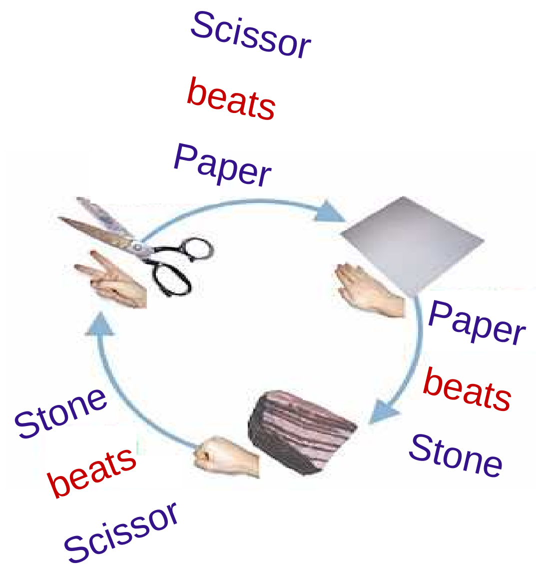

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- If selections are the same, the game starts over;

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Relation: beats $\subseteq$ {Scissor, Paper, Stone} x {Scissor, Paper, Stone}

beats	Scissor	Paper	Stone
Scissor	False	True	False
Paper	False	False	True
Stone	True	False	False

beats ={(Scissor, Paper), (Paper, Stone), (Stone, Scissor)}

Binary Relations – Properties

Let A be a set and R a binary relation over A (that is $R \subseteq A \times A$).

- R is reflexive if for every $x \in A$ it holds that xRx.

$$\forall x: x \in A: xRx$$

That is, every element x of A is in relation R with itself.

Examples:

=, ≥ are ??? binary relations over natural numbers;

> is ??? binary relation over natural numbers;

Relation *beats* from the Scissor-Paper-Stone game is ???

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  Relation *beats* from the Scissor-Paper-Stone game is not transitive.

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- R is an equivalence relation if it is *reflexive*, *symmetric* and *transitive*.

Examples: Are =, >, ≥, *beats* equivalence relations?

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Binary Relations

Let A be a set and $R \subseteq A \times A$ an *equivalence relation*.

The set of all elements y such that xRy

- is called the equivalence class of x,
- and is denoted by $[x]_R$.

$$[x]_R = \underbrace{\{y \mid xRy\}}$$

denotes "the set of all y such that xRy".

Examples: $[1]_= = ???$

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Examples: $[1]_= = \{1\}$

Binary Relations – Example

Consider the relation $\equiv_5$ over the integer numbers $\mathbb{Z}$ defined as

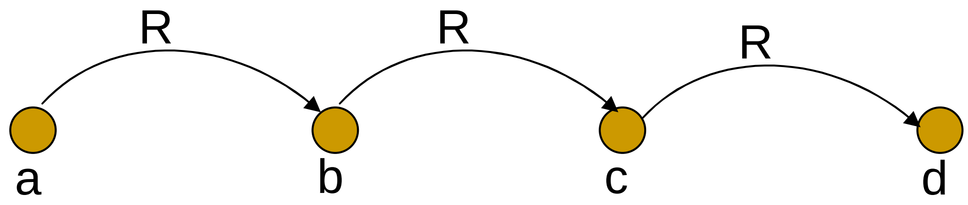
$i \equiv_5 j$ *if and only if* i-j is a multiple of 5. (where $i,j \in \mathbb{Z}$)

Is $\equiv_5$ an equivalence relation?

If so, what is $[1]_{\equiv_5}$?

Binary Relations

Let A={a,b,c,d} be a set and R⊆ A x A the relation below:



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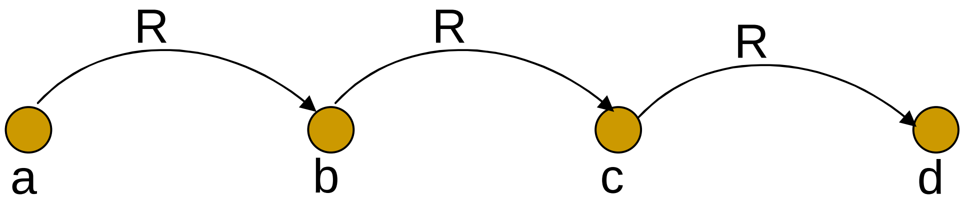
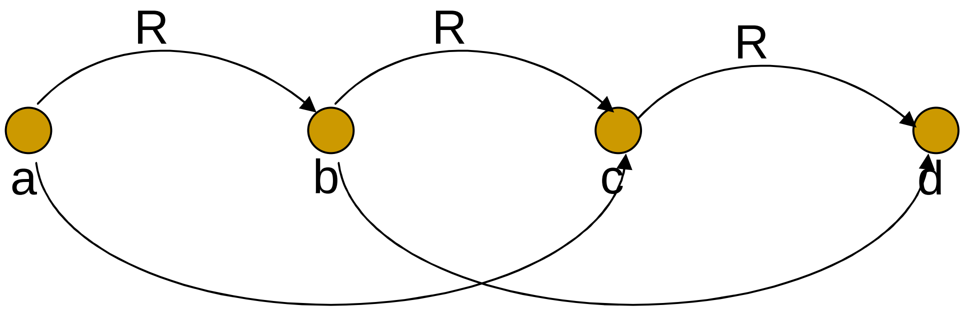


R with "1 jump":



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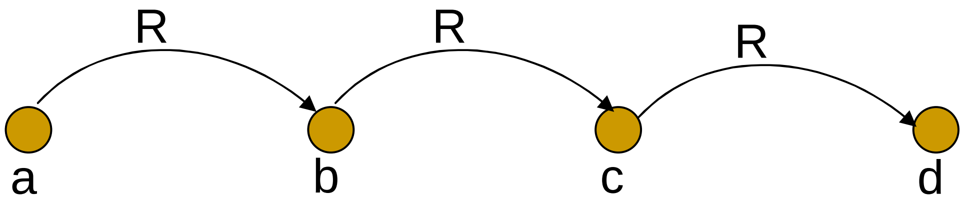

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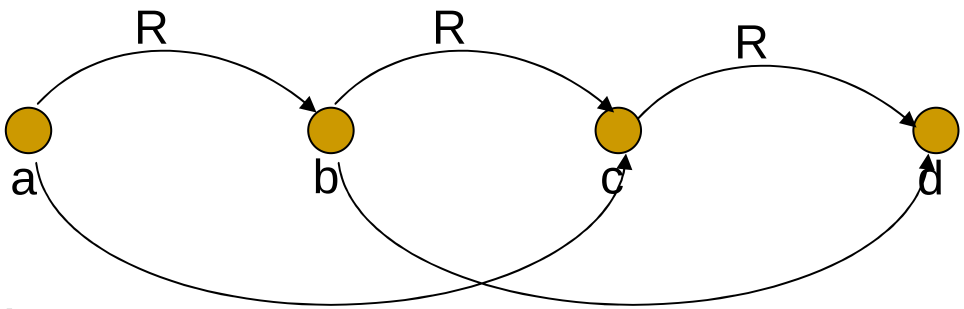
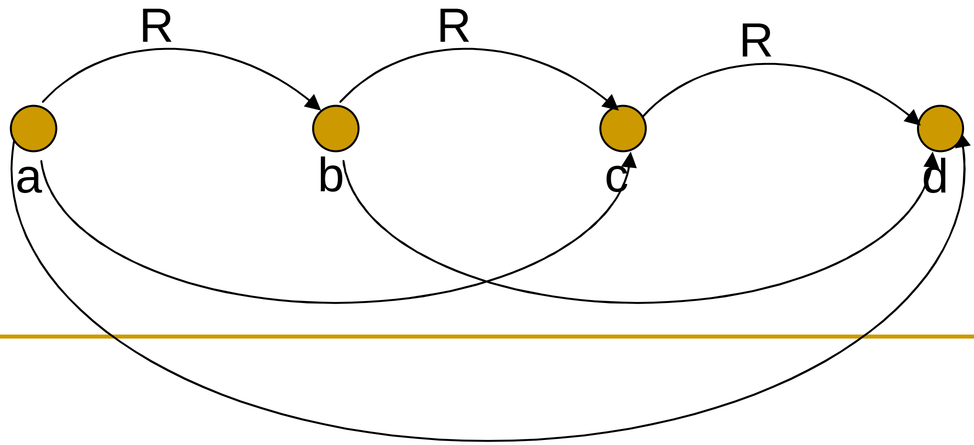


R with “2 jumps”:



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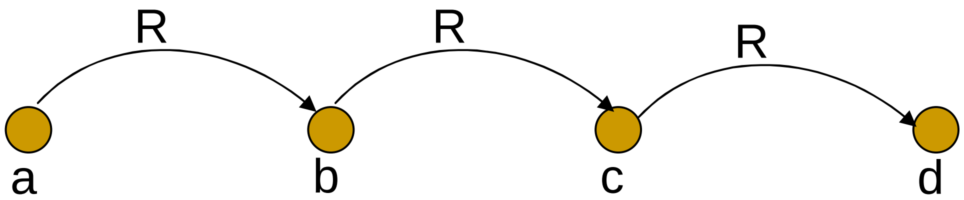

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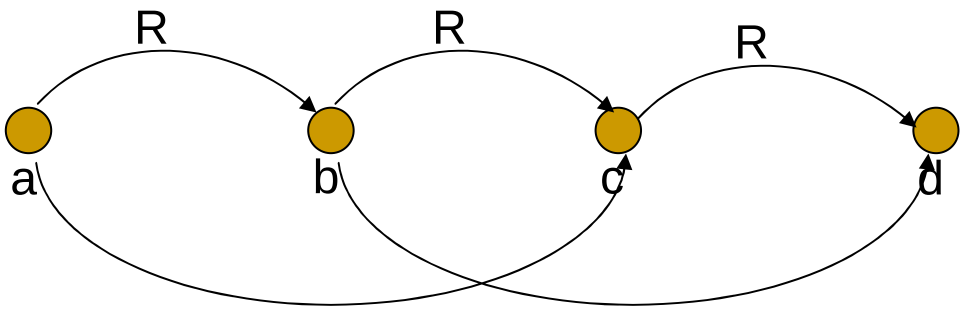
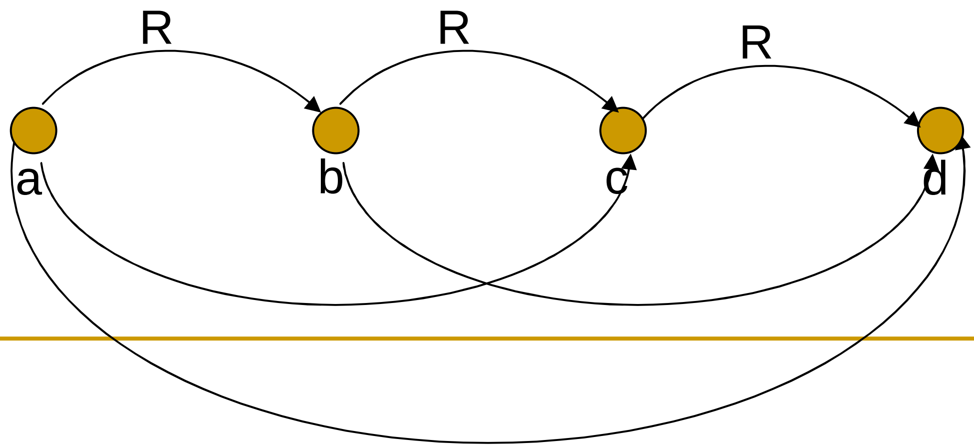
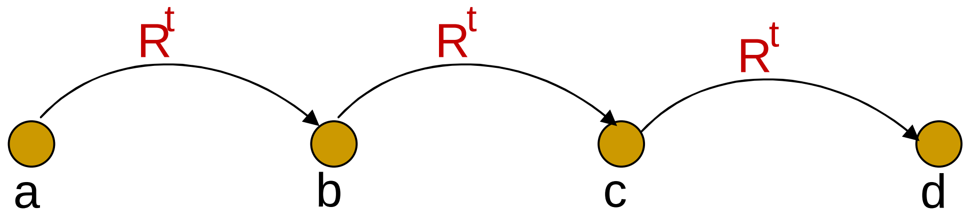


R with “2 jumps”, “3 jumps”, …:



Binary Relations – Transitive Closure/Hull (Transitive Hülle)

R^t is the transitive closure of R:



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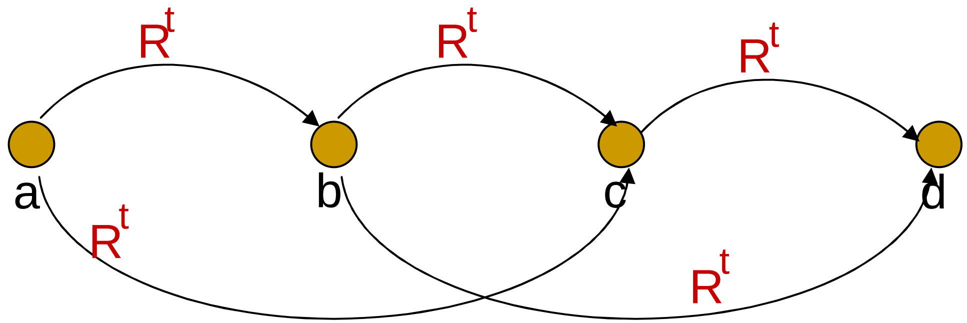

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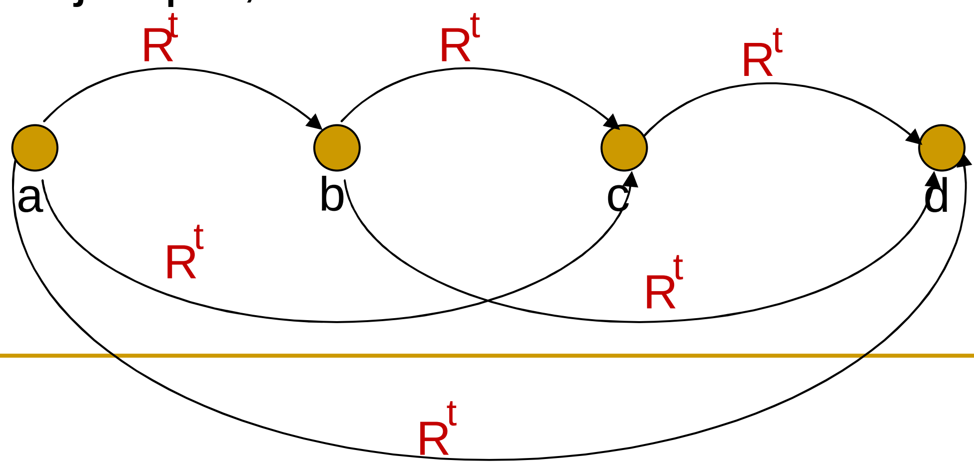

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Let A be a set and $R \subseteq A \times A$ a transitive relation.

The transitive closure of R is (the smallest) relation R^t such that

- R^t contains R: $R \subseteq R^t$;
- it extends R by all those other (indirect) relations among elements that can be obtained using the transitivity of R.

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Computing R^t:

- $R^1 = R$;
- $R^i = R^{i-1} \cup \{(a,b) \mid \exists c{::}\ (a,c) \in R^{i-1} \wedge (c,b) \in R^{i-1}\}$, for every i>1.

$$R^t = \cup_{i \geq 1} R^i = R^1 \cup R^2 \cup R^3 \cup \ldots$$

Binary Relations – Properties

Let A be a set and $R \subseteq A \times A$.

- R is irreflexive if for every $x \in A$ it holds $\neg(xRx)$.

$$\forall x: x \in A: \neg(xRx)$$

That is, no element x of A is in relation R with itself.

Examples:

> is ???

Relation *beats* from the Scissor-Paper-Stone game is ???

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  Examples:

  Are $\geq$, $=$, $\subseteq$ antisymmetric?

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- R is asymmetric if for every $x,y \in A$ it holds that *if* xRy *then* $\neg(yRx)$.

  $$\forall x,y: x,y \in A: xRy \Rightarrow \neg(yRx)$$

  That is, xRy and yRx cannot hold at the same time.

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R is asymmetric *if and only if* R is antisymmetric and irreflexive.

Binary Relations – Properties

Let A be a set and $R \subseteq A \times A$.

- R is non-symmetric (unsymmetrisch) if it is not symmetric.

  $$\forall x,y: x,y \in A: (xRy) \wedge \neg(yRx)$$

- R is a total relation if for every $x,y \in A$ either xRy or yRx holds.

  $$\forall x,y: x,y \in A: xRy \vee yRx$$

  That is, R is defined on the entire A.

  Note: Total relations are reflexive.

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  Are ≥, =, *beats* total?

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Binary Relations – Properties

Let A be a set and $R \subseteq A \times A$.

- R is acyclic (azyklisch) if there is no $x_1, x_2, \ldots, x_n \in A$ such that

  $$x_1Rx_2 \wedge x_2Rx_3 \wedge \ldots \wedge x_{n-1}Rx_n \wedge x_nRx_1 \text{ holds.}$$

$\forall n: n \in \mathbb{N}:$

$$\left(\neg(\exists x_1, x_2, \ldots, x_n: x_1, x_2, \ldots, x_n \in A: x_1Rx_2 \wedge x_2Rx_3 \wedge \ldots \wedge x_{n-1}Rx_n \wedge x_nRx_1)\right)$$

Note: Acyclic relations are irreflexive.

Example: > is acyclic.

Binary Relations – Properties

Let A be a set and $R \subseteq A \times A$.

- R is called a partial order (Halbordung, partiale Ordung) if
  - R is reflexive;
  - R is transitive;
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Example: $\geq$ is a partial order over $\mathbb{N}$.

Division / is a partial order over $\mathbb{N}$.

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- R is called a total order or a linear order (lineare/totale Ordnung) if
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  Division / is not a total order over $\mathbb{N}$.

- R is called a strict partial order (strenge Halbordnung) if
  - R is irreflexive;
  - R is transitive.

Example: > is a strict partial order over $\mathbb{N}$.

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Let A be a set and R $\subseteq$ A x A a partial order.

- An element y$\in$A is an upper bound of a set X$\subseteq$A if:
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- An element y$\in$A is an upper bound of a set X$\subseteq$A if:
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- An element y$\in$A is a least upper bound of a set X$\subseteq$A if:
  - y is an upper bound of X;
  - yRy' for all upper bounds y' of X.

Note: By antisymmetry, if y and y' are least upper bounds, then y=y'.
Hence, X has a unique least upper bound y, and we write y=lub(X).

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